$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccccccc}
 & 1 & 1 & & & & 1 \\
1 & & 1 & & & & \\
1 & 1 & & 1 & 1 & & \\
 & & 1 & & 1 & 1 & 1 \\
 & & 1 & 1 & & 1 & 1 \\
 & & & 1 & 1 & & 1 \\
1 & & & 1 & 1 & 1 &
\end{array}\right]
\end{array}
$$

which in this case is also the S matrix. Then two-step connections are given by X^2:

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccccccc}
3 & 1 & 1 & 2 & 2 & 1 & 0 \\
1 & 2 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 4 & 1 & 1 & 2 & 3 \\
2 & 1 & 1 & 4 & 3 & 2 & 2 \\
2 & 1 & 1 & 3 & 4 & 2 & 2 \\
1 & 0 & 2 & 2 & 2 & 3 & 2 \\
0 & 1 & 3 & 2 & 2 & 2 & 4
\end{array}\right]
\end{array}
$$

and the three-step ones by X^3:

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccccccc}
2 & 4 & 8 & 4 & 4 & 4 & 8 \\
4 & 2 & 5 & 3 & 3 & 3 & 3 \\
8 & 5 & 4 & 10 & 10 & 5 & 5 \\
4 & 3 & 10 & 8 & 9 & 9 & 11 \\
4 & 3 & 10 & 9 & 8 & 9 & 11 \\
4 & 3 & 5 & 9 & 9 & 6 & 8 \\
8 & 3 & 5 & 11 & 11 & 8 & 6
\end{array}\right]
\end{array}
$$

From the former, the two connections $1 \overset{(2)}{<=>} 7$ and $2 \overset{(2)}{<=>} 6$ cannot be realized because $x_{17}{}^{(2)} = x_{71}{}^{(2)} = 0$ and $x_{26}{}^{(2)} = x_{62}{}^{(2)} = 0$ (§3.01). The contacts are possible in three steps, however, since X^3 is completely filled. Thus two steps are sufficient for most contacts and three steps for all.

In determining the number of paths between two points it is desirable to eliminate redundant paths. For two-step communication this is done by deleting the main diagonal of X^2. The remaining terms represent the number of two-step paths between the stations indicated. The matrix of redundancies for three-step communication is given by $R^{(3)} = XR^{(2)} + R^{(2)}X + E^{(3)} - S$ (§3.06), which works out to be:

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} &
\begin{bmatrix}
2 & 4 & 6 & & & & 6 \\
4 & 2 & 5 & & & & \\
6 & 5 & 4 & 7 & 7 & & \\
 & & 7 & 8 & 7 & 6 & 7 \\
 & & 7 & 7 & 8 & 6 & 7 \\
 & & & 6 & 6 & 5 & 6 \\
6 & & & 7 & 7 & 6 & 6
\end{bmatrix}
\end{array}
$$

The matrix of non-redundant three-step communication paths is $X^3 - R^{(3)}$:

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} &
\begin{bmatrix}
 & & 2 & 4 & 4 & 4 & 2 \\
 & & & 3 & 3 & 3 & 3 \\
2 & & & 3 & 3 & 5 & 5 \\
4 & 3 & 3 & & 2 & 3 & 4 \\
4 & 3 & 3 & 2 & & 3 & 4 \\
4 & 3 & 5 & 3 & 3 & & 2 \\
2 & 3 & 5 & 4 & 4 & 2 &
\end{bmatrix}
\end{array}
$$

We notice that the three-step paths between 1 and 2 and 2 and 3 are all redundant but that there are two-step paths for these combinations. All other combinations have at least two three-step paths joining them.

5. *Mathematical Theory*

5.01 To carry out the following mathematical formulation and the proofs of theorems it is convenient to use some of the symbolism and nomenclature of point set theory. As there is some diversity in the literature, the symbols used are:

Sets are either defined by enumeration or by properties of the elements of the set in the form: symbol for the set [symbols used for elements of the set | defining properties of these elements]. When a single element i is treated as a set it will be denoted by (i), otherwise sets will be denoted by upper case Greek letters.

The intersection of (elements common to) two sets Γ and Φ is denoted by $\Gamma \cdot \Phi$.

The union of two sets Γ and Φ (elements contained in either or both) is denoted by $\Gamma + \Phi$. The context will make it clear whether the symbol $+$ refers to addition, matrix addition, or union.

The inclusion of a set Γ in another set Φ (all elements of Γ are elements of Φ) is denoted by $\Gamma < \Phi$. The negation is $\Gamma <^* \Phi$.

If $\Phi < \Xi$, then the complement of Φ with respect to Ξ, Φ', is defined by $\Phi + \Phi' = \Xi$ and $\Phi \cdot \Phi' = 0$ where 0 is the null set.

The inclusion of a single element i in a set Φ is denoted by $i \varepsilon \Phi$.

For any two elements i and j of a set Ξ and a subset Ω of Ξ:

$(i) + (j) < \Omega$ if and only if $i \varepsilon \Omega$ and $j \varepsilon \Omega$.
$(i) + (j) <^* \Omega$ implies $i \varepsilon \Omega'$ and/or $j \varepsilon \Omega'$.

The symbol $\delta_{ij} \begin{array}{ll} = 1 & \text{if} \quad i = j \\ = 0 & \text{if} \quad i \neq j \end{array}$

5.02. Consider a finite set Ξ of x elements denoted by $1, 2, \cdots, i, \cdots, j, \cdots, x$ for which there is defined a relationship $=>$ between elements and its negation $\neq>$ having the properties:

1. Either $i => j$ or $i \neq> j$ for all i and $j \varepsilon \Xi$.
2. $i \neq> i$.

Let a number x_{ij} be associated with i and j such that

$$x_{ij} \begin{array}{ll} = 1 & \text{if} \quad i => j \\ = 0 & \text{if} \quad i \neq> j. \end{array}$$

A matrix $X = [x_{ij}]$ is formed from the numbers x_{ij}. It will be found useful to denote the i,j entry of the n^{th} power of X, X^n, by $x_{ij}^{(n)}$.

A *symmetry* is said to exist between i and j if and only if $i => j$ and $j => i$, in which case we may write $i <=> j$. For the matrix X this requires that $x_{ij} = x_{ji} = 1$. If, however, either $i => j$ and $j \neq> i$ or $i \neq> j$ and $j => i$ then an *antimetry* is said to exist between i and j.

The *symmetric matrix* S associated with the matrix X is defined by $S = [s_{ij}]$, where

$$s_{ij} = s_{ji} \begin{array}{ll} = 1 & \text{if} \quad x_{ij} = x_{ji} = 1, \quad \text{i.e.,} \quad i <=> j. \\ = 0 & \text{otherwise.} \end{array}$$

The i,j entry of the n^{th} power of S is $s_{ij}^{(n)}$.

5.03. Definitions:

1. An ordered sequence with $n+1$ members, $i \equiv \gamma_1, \gamma_2, \cdots, \gamma_n, \gamma_{n+1} \equiv j$, is an n-*chain* Γ from i to j if and only if

$$i \equiv \gamma_1 => \gamma_2, \gamma_2 => \gamma_3, \cdots, \gamma_n => \gamma_{n+1} \equiv j.$$

In brief, $i \overset{(n)}{=>} j$ indicates that there exists an n-chain from i to j, which may also be enumerated as $i \equiv \gamma_1, \gamma_2, \cdots, \gamma_n, \gamma_{n+1} \equiv j$, or, when no ambiguity will arise, as $i, k, l, \cdots, p, q, j$ with the ordering being indicated by the written order of the sequence.

2. Two n-*chains* Γ and Φ *are equal* if and only if the r^{th} member of Γ equals the r^{th} member of Φ, i.e., $\gamma_r = \phi_r$, for $1 \leqslant r \leqslant n+1$.

If this is not true, then Γ and Φ are *distinct*.

3. Each pair of elements γ_k and γ_m of an n-chain with $1 \leqslant k < m \leqslant n+1$ and $\gamma_k = \gamma_m$ is said to be the *redundant pair* (k,m). An n-chain is *redundant* if and only if it contains at least one redundant pair.

4. The elements $1, 2, \cdots, t$ $(t \geqslant 3)$ form a *clique* Θ of t members if and only if each element of Θ is symmetric with each other element of Θ, and there is no element not in Θ symmetric with all elements of Θ.

This is equivalent to

$x_{ij} = 1 - \delta_{ij}$ for $i, j = 1, 2, \cdots, t$ but not for $i, j = 1, 2, \cdots, t, t+1$, whatever the $(t+1)^{st}$ element.

5.04. Theorem 1: $x_{ij}{}^{(n)} = c$ if and only if there exist c distinct n-chains from i to j.
Proof: By definition of matrix multiplication

$$x_{ij}{}^{(n)} = \sum_{k \varepsilon \Xi} \cdots \sum_{q \varepsilon \Xi} x_{ik} x_{kl} \cdots x_{pq} x_{qj},$$

with the summations over n—1 indices. Suppose that the indices have been selected such that $i, k, l, \cdots, p, q, j$ is an n-chain from i to j.

Then by definition 1 (§5.03)

$$x_{ik} = x_{kl} = \cdots = x_{pq} = x_{qj} = 1,$$

and if the indices were not so selected then at least one $x_{rs} = 0$. Thus n-chains contribute 1 to the sum and other ordered sequences contribute 0. Since the indices take on each possible combination of values just once, every distinct n-chain is represented just once. If there are c such n-chains, then there are a total of c ones in the summation.

5.05. Theorem 2: An element of Ξ has a main diagonal value of c in X^2 if and only if it is symmetric with c elements of Ξ.
Proof: Let Φ be the set of j's for which $i <=> j$. By definition

$$x_{ii}{}^{(2)} = \sum_{j \varepsilon \Phi} x_{ij} x_{ji} + \sum_{j \varepsilon \Phi'} x_{ij} x_{ji} = \sum\nolimits_1 + \sum\nolimits_2.$$

$\sum_1 = c$ by theorem 1 (§5.04) and $\sum_2 = 0$ because i and j are not symmetric for $j \varepsilon \Phi'$, so either $x_{ij} = 0$ or $x_{ji} = 0$ or both. Thus if i is symmetric with c elements of Ξ, $x_{ij}{}^{(2)} = c$.

If $x_{ii}{}^{(2)} = c$, then by theorem 1 there exist c distinct j's such that $x_{ij} = x_{ji} = 1$, i.e., $i <=> j$ for c j's.

5.06. Theorem 3: An element i is contained in a clique if and only if the i^{th} entry of the main diagonal of S^3 is positive.
Proof: Suppose that i is contained in a clique Θ.
By definition

$$s_{ii}{}^{(3)} = \sum_{(j)+(k)<\Xi} \sum s_{ij} s_{jk} s_{ki}.$$

Select j and k such that $(j) + (k) < \Theta$ and such that $i \neq j \neq k \neq i$. Such elements exist by the definition of a clique (definition 4, §5.03). It is true by the definition of a clique and of the matrix S that: $s_{ij} = s_{ji} = s_{jk} = s_{kj} = s_{ik} = s_{ki} = 1$ for such j and k. Thus this choice of j and k contributes 2 to the summation, and because $s_{ij} \geqslant 0$ for all i and j there are no negative contributions to the sum; therefore $s_{ii}{}^{(3)} \geqslant 2 > 0$.

Suppose that $s_{ii}{}^{(3)} > 0$. Then there exists at least one pair of elements of j and k such that $s_{ij} = s_{jk} = s_{ki} = 1$ and this implies $i <=> j$, $j <=> k$, and $k <=> i$. If there are no other elements symmetric with i, j, and k then these three form a clique. If there is another element symmetric with these three, then consider the set of four formed by adding it to the previous three. If there is no other element symmetric with these four, they form a clique. If there is, add it to the set and continue the process. Since the set Ξ contains only a finite number of elements, the process must terminate giving a clique containing i.

5.07. Theorem 4: If 1) Θ_σ are cliques of t_σ members, 2) the sets $\Delta_\nu = \Theta_\nu \cdot (\Theta_1 + \Theta_2 + \cdots + \Theta_{\nu-1})$ have d_ν members, and 3) i is contained in the cliques Θ_σ, $\sigma = 1, 2, \cdots, m$, then

$$s_{ii}{}^{(3)} = \sum_{\sigma=1}^{m} \{(t_\sigma - 2)(t_\sigma - 1) - (d_\sigma - 2)(d_\sigma - 1)\} + 2.$$

Proof: By definition

$$s_{ii}{}^{(3)} = \sum_{(j)+(k)<\Xi} \sum s_{ij} s_{jk} s_{ki}.$$

The set of all the pairs j, k is the union of the following three mutually exclusive sets:

Ψ_1 $[j, k \mid$ there exists ν such that $(j) + (k) < \Theta_\nu$; there does not exist α such that $(j) + (k) < \Theta_\alpha$, $(j) + (k) <^* \Delta_\alpha]$

Ψ_2 $[j, k \mid$ there does not exist α such that $(j) + (k) < \Theta_\alpha]$

Ψ_3 $[j, k \mid$ there exists α such that $(j) + (k) < \Theta_\alpha$, $(j) + (k) <^* \Delta_\alpha]$.

1. For Ψ_1 then either

a) $(j) + (k) <^* \Theta_\alpha$ for all α. This is not possible because $(j) + (k) < \Theta_\nu$;

b) $(j) + (k) < \Delta_\alpha$ for all α. This is not possible because $\Delta_1 = 0$;

or c) $(j) + (k) < \Theta_\alpha$ if and only if $(j) + (k) < \Delta_\alpha$ for all α. This is not possible because $\Delta_1 = 0$. Thus Ψ_1 is empty.

2. $(j) + (k) < \Psi_2$ implies $s_{ij}s_{jk}s_{ki} = 0$ for $s_{ij}s_{jk}s_{ki} = 1$ implies that i, j, and k are either a clique or a subset of a clique (by the argument of theorem 3), but $(j) + (k) < \Psi_2$ implies j and k are not contained in any clique.

3. Ψ_3 gives that

$$s_{ii}{}^{(3)} = \sum_{(j)+(k)<\Psi_3}\sum s_{ij}s_{jk}s_{ki}$$

$$= \sum_{\nu=1}^{m} \left\{ \sum_{\substack{(j)+(k)<\Theta_\nu \\ (j)+(k)<^*\Delta_\nu}}\sum s_{ij}s_{jk}s_{ki} \right\}.$$

We observe that: $\Omega_1[j, k \mid (j) + (k) < \Theta_\nu] = \Omega_2[j, k \mid (j) + (k) < \Theta_\nu, (j) + (k) < \Delta_\nu] + \Omega_3[j, k \mid (j) + (k) < \Theta_\nu, (j) + (k) <^* \Delta_\nu]$ and since $\Omega_2 \cdot \Omega_3 = 0$, it follows that $\sum_{\Omega_1} = \sum_{\Omega_2} + \sum_{\Omega_3}$ or $\sum_{\Omega_3} = \sum_{\Omega_1} - \sum_{\Omega_2}$. Ω_1 is the set of all ordered pairs $(j) + (k) < \Theta_\nu$. If $i \neq j \neq k \neq i$, then $s_{ij} = s_{jk} = s_{ki} = 1$, otherwise one of the $s_{pq} = 0$. Since every Θ_ν contains t_ν elements, there are ${}_{t_\nu-1}P_2$ ordered pairs satisfying these conditions. Thus:

$$\sum_{\Omega_1} = {}_{t_\nu-1}P_2 = (t_\nu - 2)(t_\nu - 1).$$

Similarly

$$\sum_{\Omega_2} = \begin{cases} (d_\nu - 2)(d_\nu - 1), & \nu > 1 \\ 0 & \nu = 1 \quad \text{since} \quad \Delta_1 = 0. \end{cases}$$

Combining these,

$$\sum_{\Omega_3} = \begin{cases} (t_\nu - 2)(t_\nu - 1) - (d_\nu - 2)(d_\nu - 1), & \nu > 1 \\ (t_\nu - 2)(t_\nu - 1) & , \nu = 1. \end{cases}$$

Summing over ν gives

$$s_{ii}^{(3)} = \sum_{\nu=2}^{m} \{ (t_\nu - 2)(t_\nu - 1) - (d_\nu - 2)(d_\nu - 1) \}$$
$$+ (t_1 - 2)(t_1 - 1)$$
$$= \sum_{\nu=1}^{m} \{ (t_\nu - 2)(t_\nu - 1) - (d_\nu - 2)(d_\nu - 1) \} + 2.$$

Since the entries $s_{ij}^{(3)}$ are uniquely determined from the entries of S by the laws of matrix multiplication, all valid methods of calculating $s_{ii}^{(3)}$ will give the same result. Specifically, in the above formula the numbering of the cliques is immaterial.

Similar formulas to that just deduced may be given for the off-diagonal terms of S^3, but they are considerably more complex, and, to date, they have not been found useful in applications.

5.08. Theorem 5: If 1) Θ is a set of t members with $t \geqslant 3$, 2) $s_{ii}^{(3)} = (t-2)(t-1)$ for i contained in Θ, and 3) $s_{jj}^{(3)} = 0$ for j contained in Θ', then Θ is a clique of t members.
Proof: There are two cases:

1. $i <=> j$ for all $i, j \;\varepsilon\; \Theta$, then Θ is a clique by definition 4 (§5.03) and theorem 3 (§5.06), and it has t members by part 1 of the hypothesis.

2. There exist p and $q \;\varepsilon\; \Theta$ such that p and q are not symmetric. Then by definition

$$S_{ii}^{(3)} = \sum_{(j)+(k)<\Theta} \sum s_{ij}s_{jk}s_{ki}$$
$$+ \sum_{(j)+(k)<^*\Theta} \sum s_{ij}s_{jk}s_{ki}.$$

If $s_{ij}s_{jk}s_{ki} = 1$, the elements i, j, and k are a clique or a subset of a clique and thus by hypothesis (3) and theorem 3 (§5.06) they are all contained in Θ; therefore the second sum $= 0$. Introduce in Ξ sufficient relationships $p => q$ to make Θ a clique Φ of t members. Since $s_{ij} \geqslant 0$ for all i and j, the introduction of these $s_{pq} = 1$ must increase the sum by 2 or more, for at least two additional 3-chains are introduced (i,p,q,i and i,q,p,i); hence by theorem 4 (§5.07)

$$S_{ii}^{(3)} = \sum_{(j)+(k)<\Phi} \sum s_{ij}s_{jk}s_{ki} - 2 = (t-2)(t-1) - 2$$
$$< (t-2)(t-1),$$

which is contrary to hypothesis (2). Therefore Θ is a clique of t members.

5.09. Redundancies:

By definition 3 (§5.03) an n-chain is redundant if and only if it contains at least one redundant pair (k,m), where a redundant pair defines two members of the n-chain γ_k and γ_m with $\gamma_k = \gamma_m$ and $k < m$. If these ordered subscript pairs (k,m) and the end point pair (i,j) (the latter not necessarily a redundant pair) are considered as sets, then five classes of mutually exclusive redundant n-chains may be defined which include all redundant n-chains:

1. The A_n class: There exists at least one redundant pair (k,m) and it has the property:

$$(k,m) \cdot (i,j) = 0 .$$

2. The B_n class: There exists one and only one redundant pair (k,m) and it has the property:

$$(k,m) \cdot (i,j) = i .$$

3. The C_n class: There exists one and only one redundant pair (k,m) and it has the property:

$$(k,m) \cdot (i,j) = j .$$

4. The D_n class: There exist two and only two redundant pairs (k,m) and (p,q) and they have the properties:

$$(k,m) \cdot (i,j) = i$$
$$(p,q) \cdot (i,j) = j .$$

5. The E_n class: There exists one and only one redundant pair (k,m) and it has the property:

$$(k,m) \cdot (i,j) = (i,j).$$

If there are t n-chains $i \overset{(n)}{=\!\!>} j$ of the class A_n from i to j, then define $a_{ij}^{(n)} = t$. From these numbers the matrix $A^{(n)} = [a_{ij}^{(n)}]$ is formed. This is the matrix of redundant n-chains of the class A_n. If $R^{(n)}$ is the matrix of redundant n-chains it follows, if analogous definitions are made for matrices of the other four classes, that

$$R^{(n)} = A^{(n)} + B^{(n)} + C^{(n)} + D^{(n)} + E^{(n)} .$$

It follows directly from the definitions and the limitations on n that

$$R^{(1)} = 0$$
$$\overline{R^{(2)}} = [\delta_{ij} x_{ij}{}^{(2)}] = E^{(2)}$$
$$A^{(3)} = 0 \,.$$

It will now be proved that $D^{(3)} = S$. By the definition of the class D_3, there exist two and only two redundant pairs (k,m) and (p,q), and they have the properties:

$$(k,m) \cdot (i,j) = i$$
$$(p,q) \cdot (i,j) = j \,.$$

These pairs may define in total either three or four members of the 3-chain (three members when $m = p$, but no fewer for if $k = p$ and $m = q$ then $(k,m) \cdot (i,j) = (i,j)$, which is contrary to the definition of D_3). Suppose $m = p$, then either $i = \gamma_2 = j$ or $i = \gamma_3 = j$, which is impossible for $i \neq > i$ by assumption. Thus $m \neq p$. With four members there are two possibilities for a redundant 3-chain: either $i = \gamma_2$, $\gamma_3 = j$ or $i = \gamma_3$, $\gamma_2 = j$. The former is impossible by the previous argument; thus the only 3-chains of the class D_3 are of the form

$$i\,, \gamma_2\,, \gamma_3\,, j \equiv i\,, j\,, i\,, j\,;$$

that is,

$$d_{ij}{}^{(3)} \begin{array}{ll} = 1 & \text{if} \quad i <=> j \\ = 0 & \text{otherwise.} \end{array}$$

Therefore, by the definition of S, we have $D^{(3)} = S$.

If the matrices of redundancies up to and including $R^{(n-2)}$ are known, then we can find $A^{(n)}$ by $A^{(n)} = XR^{(n-2)}X$.
Proof: By the definition of the class A_n, a redundant n-chain of this class has the form

$$i \equiv \gamma_1\,, \gamma_2\,, \overset{(a)}{\text{———}}\,, \gamma_k\,, \overset{(b)}{\text{———}}\,, \gamma_m\,, \overset{(c)}{\text{———}}\,, \gamma_n\,, \gamma_{n+1} \equiv j\,,$$

where $a + b + c + 5 = n$, $k < m$, and $\gamma_k = \gamma_m$.

It follows from the definition that $\underline{p} \equiv \gamma_2 \overset{(n-2)}{\Longrightarrow} \gamma_n \equiv q$ is a redundant n—2 chain, and each such distinct n—2 chain determines no more than one distinct redundant n-chain from i to j. Thus the number of redundant n-chains of type A_n from i to j is the sum over all combinations $p \equiv \gamma_2$ and $q \equiv \gamma_n$ for the number of redundant n—2 chains from p to q, that is,

$$a_{ij}{}^{(n)} = \sum_{(p)+(q)<\equiv} \sum x_{ip} r_{pq}{}^{(n-2)} x_{qj}$$

or

$$A^{(n)} = XR^{(n-2)}X \,.$$

If the matrix $[e_{ij}{}^{(n)}]$ is defined as

$$[e_{ij}{}^{(n)}] = XR^{(n-2)}X + D^{(n)}$$

then the relations

$$\begin{aligned} A^{(n)} + B^{(n)} + D^{(n)} &= R^{(n-1)}X \\ A^{(n)} + C^{(n)} + D^{(n)} &= XR^{(n-1)} \\ E^{(n)} &= [\delta_{ij}(x_{ij}{}^{(n)} - e_{ij}{}^{(n)})] \end{aligned}$$

follow through an enumeration of cases and by using similar patterns of proof to that just given.

These various relations permit the specific conclusions:

$$\begin{aligned} R^{(2)} &= [\delta_{ij}x_{ij}{}^{(2)}] = E^{(2)} \\ R^{(3)} &= XR^{(2)} + R^{(2)}X + E^{(3)} - S \end{aligned}$$

and the general result

$$\begin{aligned} R^{(n)} = XR^{(n-1)} &+ R^{(n-1)}X - XR^{(n-2)}X \\ &+ E^{(n)} - D^{(n)} . \end{aligned}$$

This latter expression is not useful in its present form because $D^{(n)}$ has not been expressed in terms of the matrices of redundancies up to and including $R^{(n-1)}$. This problem of the determination of the matrix of redundant n-chains is left as an unsolved problem of both theoretical and practical interest.

5.10. Uniqueness:

In certain applications it is desirable to know whether a power of a matrix uniquely determines the matrix. This is not true in general, for Sylvester's theorem gives a multiplicity of n^{th} roots of a matrix. The matrices being considered are rather specialized, however, and it is possible that some degree of uniqueness may exist.

The following two theorems indicate certain sufficient conditions for uniqueness. Since these theorems do not utilize completely the special characteristics of the matrices in this study, it is probable that more appropriate theorems can be proved.

5.11. Theorem 6: If p and q are positive integers, if two integers a and b can be found such that $ap - bq = 1$, and if X is a non-singular matrix, then the powers X^p and X^q uniquely determine X. Proof: Suppose that there exist two non-singular matrices X and Y such that $X^p = Y^p$ and $X^q = Y^q$. Then $X^{ap} = Y^{ap}$ and $X^{bq} = Y^{bq}$. Now, form $X^{bq}Y = Y^{bq}Y = Y^{bq+1} = Y^{ap}$, since $ap - bq = 1$. Similarly $X^{bq}X = X^{bq+1} = X^{ap}$. But since $X^{ap} = Y^{ap}$ it follows that $X^{bq}X = X^{bq}Y$.

Since X is non-singular, $|X^{bq}| \neq 0$, and thus there exists a unique inverse of X^{bq}, X^{-bq}, such that $X^{-bq}X^{bq} = I$; therefore $X = Y$.

5.12. Theorem 7: If n is a positive odd integer and S a real symmetric matrix, then there is one and only one real symmetric n^{th} root of S.

Proof: 1. There is one such n^{th} root.

Since S is real and symmetric there exists a real orthogonal matrix P such that $P'SP = D$ (P' is the transpose of P) is diagonal with real entries d_{ii} which are the characteristic roots of S.* Assume P is so chosen that $d_{11} \leqslant d_{22} \leqslant \cdots \leqslant d_{mm}$. Let B be the diagonal matrix of the real n^{th} roots of the elements of D, i.e., $b_{ii} \underline{=} \text{real } (d_{ii})^{1/n}$, so

$$B^n = D. \tag{1}$$

Define $R \underline{=} PBP'$. Then $R^n = S$, for

$$R^n = (PBP')^n = PB^nP' = PDP' = S.$$

Since B is real and diagonal and P is real and orthogonal, R is real and symmetric.

2. There is only one such n^{th} root.

Suppose there exists a real symmetric matrix R_1 not equal to R such that $R_1^n = S$. Then there exists an orthogonal matrix Q such that $Q'R_1Q \underline{=} T$ is diagonal in the characteristic roots of R_1, and ordered as before. Consider the n^{th} power of T:

$$\begin{aligned} T^n &= (Q'R_1Q)^n = Q'R_1^nQ = Q'SQ \\ &= Q'PDP'Q = (P'Q)'D(P'Q) \\ T^n &= U'DU, \end{aligned} \tag{2}$$

where U is the orthogonal matrix $P'Q$. Since $U' = U^{-1}$, T^n and D are similar, and hence have the same characteristic roots.† Because they are diagonal in the characteristic roots, ordered in the same way, they are equal:

$$D = T^n. \tag{3}$$

Substituting (3) in (2)

$$\underline{D} = U'DU$$

or

$$U\underline{D} = DU.$$

*MacDuffee, C. C. Vectors and matrices. Ithaca, N. Y.: The Mathematical Association of America, 1943, pp. 166-170.
†Ibid., p. 113.

By definition of matrix multiplication this means

$$\sum_j u_{ij} d_{jk} = \sum_j d_{ij} u_{jk} .$$

Since D is diagonal, this reduces to

$$u_{ik} d_{kk} = d_{ii} u_{ik}$$

or

$$u_{ik} (d_{kk} - d_{ii}) = 0 . \tag{4}$$

Since the d_{kk} are real and n is odd, equation (4) implies

$$u_{ik} [(d_{kk})^{1/n} - (d_{ii})^{1/n}] = 0$$

where the $(d_{kk})^{1/n}$ are real. Thus by the definition of B ,

$$UB = BU$$

or

$$B = U'BU . \tag{5}$$

By (1) and (3)

$$T^n = D = B^n ,$$

but by construction T and B are both real diagonal matrices and n is odd, so this implies

$$T = B .$$

This substituted in (5) gives

$$T = U'BU = Q'PBP'Q$$

or

$$QTQ' = PBP'.$$

But $QTQ' = R_1$ and $PBP' = R$ by definition; therefore

$$R_1 = R .$$

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