THE LEAST-AREA TETRAHEDRAL TILE OF SPACE

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ABSTRACT. We determine the least-area unit-volume tetrahedral tile of Euclidean space, without the constraint of Gallagher *et al.* that the tiling uses only orientation-preserving images of the tile. The winner remains Sommerville's type 4v.

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1. Introduction

Gallagher et al. [GGH] showed that the least-surface-area, unit-volume, face-to-face, tetrahedral tile is Sommerville's [So2] type 4v, referred to in our paper as Sommerville No. 1 and illustrated in Figure 1, assuming that the tiling is orientation preserving (does not need to reflect the tile).

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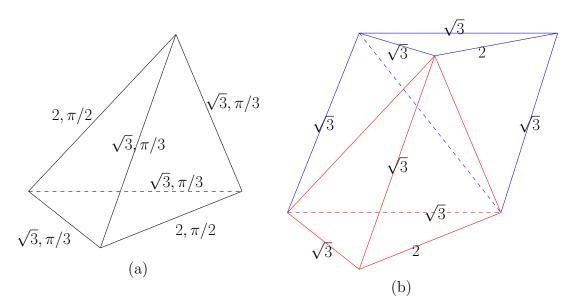


FIGURE 1. (1a): The Sommerville No. 1 tetrahedral tile labeled with edge lengths and dihedral angles. (1b): The arrangement of three copies of the Sommerville No. 1 tile into a triangular prism, proving that it tiles space.

We prove that the Sommerville No. 1 even among non-orientation-preserving tilings is still surface area minimizing.

Theorem 1.1 (Thm. 11.1). The least-area, face-to-face tetrahedral tile of unit volume is uniquely the Sommerville No. 1.

The least-surface-area property in question could be interpreted as being the closest one to the regular tetrahedron among all face-to-face space-tiling tetrahedra.

Currently, there is no list of all tetrahedral tiles. Moreover, there are infinitely many tetrahedra that tile in a non-orientation-preserving manner, all known examples provided by Goldberg [Gol]. We characterize a subclass of tetrahedra that includes all possible candidates for surface area minimization and find that there are none that beat the Sommerville No. 1. First, we classify all tetrahedra into 25 types (Sect.

6), fifteen with the tiles completely identified. The Sommerville No. 1 is the least-area tile among these fifteen types. The remaining ten types can be reduced to a total of 2074 cases by certain linear equations and bounds on the dihedral angles (Sects. 7–9). In particular, every dihedral angle must be greater than 36.5 degrees (Cor. 8.3). Necessary conditions for dihedral angles of tetrahedra of the given type, implemented by careful and rigorous computer code, reduce the cases from 2074 to seven (Sect. 9). These seven cases are eliminated by special arguments (Sect. 10), leaving no candidate to tile with less surface area than the Sommerville No. 1.

Trivial Dehn invariant is a necessary but not sufficient condition for a polyhedron to tile and is therefore a potentially useful tool in narrowing down to potential tiles. Unfortunately, we were not able to find a useful way to compute Dehn invariants.

As a consequence of searching for the surface-area-minimizing tetrahedral tile without a complete list of tetrahedral tiles, this paper also completely characterizes the tiles of fifteen of 25 types and provides some necessary conditions for the remaining ten types to tile. This paper is only concerned with surface area minimization, but those interested in tetrahedral tiles in general may find this information useful. The known properties of all 25 types are given in Sections 6–7. Many of the methods used here, particularly the edge-length graphs described in Section 4, could be extended to tiles of other *n*-hedra or non-face-to-face tiles (Section 11).

Even the characterization of convex planar tiles was settled only recently with Rao's [Rao] characterization of pentagonal tilings.

As for general n-hedral tiles, Gallagher $et\ al.$ [GGH] give conjectures for all values of n. The only previous proofs were for the cube (n=6) and the triangular prism (n=5). In both cases, the result for the least-area tile is simply the least-area n-hedron, which happens to tile. This paper presents the first proven case where requiring an n-hedron to tile changes the surface area minimizer.

Outline of paper. Section 2 sets terminology and summarizes previouslyknown facts about tetrahedra that are used in proofs throughout this paper. Section 3 gives a comprehensive review of known tetrahedral tiles and proves that the Sommerville No. 1 is the least-area tile among known tetrahedral tiles.

Section 4 introduces a combinatorial object called an edge-length graph, which is used to deduce linear systems of dihedral angles that are necessary conditions for tetrahedral tiles.

Section 5 completely identifies all tiling tetrahedra whose dihedral angles are all of the form $2\pi/n$.

Section 6 classifies all tetrahedra into 25 types. Of these 25 types, the tiling behavior of eleven types has been completely characterized by previous literature. An additional four types are proven not to tile, which was not previously addressed by the literature. Some necessary conditions for the remaining ten types to tile are given in Section 7.

Section 8 provides a lower bound on dihedral angles that allows us to search the remaining ten types for a tetrahedral tile with less surface area than the Sommerville No. 1.

Section 9 describes how computer code is used with necessary conditions from Sections 7 and 8 to reduce the number of candidates to beat the Sommerville No. 1 to seven remaining cases. proves that none of these remaining seven cases tiles with less surface area than the Sommerville No. 1.

Finally, in Section 11 we conclude that the Sommerville No. 1 is the least-surface-area tetrahedral tile. We comment on how methods used in this paper may extend to tilings of other polyhedra or non-face-toface-tiles. We conjecture that the list of tetrahedral tiles identified in previous literature is exhaustive. We also conjecture that a surfacearea-minimizing n-hedron is identical to its mirror image.

Note on code. Many proofs in this paper are computer-assisted. The code for these computations is available in the GitHub repository at https://github.com/arjunkakkar8/Tetrahedra.

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2. Tetrahedra and Tilings

Section 2 sets terminology and summarizes previously known facts about tetrahedra that are used in proofs throughout this paper.

We use notation consistent with previous literature. Let $T = V_1V_2V_3V_4$ be a tetrahedron where V_1, V_2, V_3, V_4 are vertices in \mathbb{R}^3 . For $\{i, j, k, l\} = \{1, 2, 3, 4\}$, let F_i be the face opposite the vertex V_i with area denoted $|F_i|$. Let e_{ij} be the edge joining V_i and V_j with length denoted $d_{ij} = |e_{ij}|$. (Note that the edge between the faces F_i and F_j is e_{kl} , not e_{ij} .) Let θ_{ij} be the dihedral angle between two faces adjacent to the edge e_{ij} , i.e. between faces F_k and F_l .

We assume that all tilings are face-to-face. We make a distinction between orientation-preserving and non-orientation-preserving tilings.

Definition 2.1. A tiling is *orientation-preserving* if any two tiles are equivalent under an orientation-preserving isometry of \mathbb{R}^3 .

If a tetrahedron tiles without the use of its mirror image or if the tetrahedron is not distinct from its mirror image, then the resulting tiling is orientation-preserving. On the other hand, if a tetrahedron tiles with the use of its distinct mirror image, then the resulting tiling is non-orientation-preserving.

Lemmas 2.2–2.6 summarize previously known and useful facts.

Lemma 2.2 (Law of cosines for a tetrahedron).

$$|F_1|^2 = |F_2|^2 + |F_3|^2 + |F_4|^2 - 2(|F_2||F_3|\cos\theta_{14} + |F_2||F_4|\cos\theta_{13} + |F_3||F_4|\cos\theta_{12}).$$

Lemma 2.3 relates the areas of the faces to the dihedral angles between them. Geometrically, it says that the area of the base face F_1 is equal to the sum of the areas of the projections of the remaining faces.

Lemma 2.3.

$$|F_1| = |F_2| \cos \theta_{34} + |F_3| \cos \theta_{24} + |F_4| \cos \theta_{23}.$$

Moreover, if $c_{ij} = \cos \theta_{ij}$, then:

$$\begin{vmatrix} -1 & c_{34} & c_{24} & c_{23} \\ c_{34} & -1 & c_{14} & c_{13} \\ c_{24} & c_{14} & -1 & c_{12} \\ c_{23} & c_{13} & c_{12} & -1 \end{vmatrix} = 0,$$

and the area vector $(|F_1|, |F_2|, |F_3|, |F_4|)^T$ is in the null space of the matrix corresponding to this determinant.

The following lemmas from Wirth and Dreiding [WD2] impose constraints on the dihedral angles and edge lengths of a tetrahedron.

Lemma 2.4 ([WD2, Lemmas 1 and 3]).

$$\Omega_1 = \theta_{12} + \theta_{13} + \theta_{14} - \pi > 0,$$

where Ω_1 is the solid angle at the vertex V_1 and

$$\theta_{13} + \theta_{14} + \theta_{24} + \theta_{23} < 2\pi$$
.

Lemma 2.5 ([WD2, Lemma 4 and Rmk. 3]). A sextuple $(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34})$ provides the edge lengths of a tetrahedron if and only if

- (1) For any distinct $i, j, k \in \{1, 2, 3, 4\}$, the triangle inequality $d_{ij} < d_{jk} + d_{ik}$ holds, where we interpret $d_{ij} = d_{ji}$, and
- (2) the Cayley-Menger determinant is positive:

$$D = \begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & 1 \\ d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 & 1 \\ d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 & 1 \\ d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} > 0.$$

Moreover, the volume V of the tetrahedron can be written in terms of this determinant as

$$D = 288V^2.$$

Lemma 2.6 ([WD2, Theorem 1]). The dihedral angle θ_{ij} is given by

$$\cos \theta_{ij} = \frac{D_{ij}}{\sqrt{D_{ijk}D_{ijl}}},$$

where

$$D_{ij} = -d_{ij}^4 + \left(d_{ik}^2 + d_{il}^2 + d_{jk}^2 + d_{jl}^2 - 2d_{kl}^2\right)d_{ij}^2 + \left(d_{ik}^2 - d_{jk}^2\right)\left(d_{jl}^2 - d_{il}^2\right)$$

and

$$D_{ijk} = -16 \left| F_l \right|^2.$$

The next proposition says that the dihedral angles of a tetrahedron determine the tetrahedron.

Proposition 2.7. The dihedral angles (order matters) determine the tetrahedron up to translation, rotation, and scaling.

Proof. A tetrahedron is made up of four planes. Specifying the dihedral angles means that the angles between all pairs of planes are specified.

Consider the incomplete tetrahedron with three planes intersecting at a vertex at three specified dihedral angles. This configuration is uniquely determined up to translation and rotation. Then, the fourth plane has a fixed orientation with respect to the incomplete tetrahedron, and its translation corresponds to scaling the tetrahedron. Hence, the tetrahedron is fixed up to translation, rotation and scaling.

3. Previous Literature

Section 3 summarizes previous results on tetrahedral tiles.

Sommerville. Sommerville [So1] provided a list of every orientation-preserving face-to-face tetrahedral tile, which was proven to be complete by Edmonds [Ed]. In contrast, our definition of space-filling also includes non-orientation-preserving tiles. The tetrahedra Sommerville dismissed because of being non-orientation preserving are considered in this paper, but they are not labeled by Sommerville's nomenclature. A full description of the types considered in this paper is given in Section 6.

Out of eleven total candidates, Sommerville proved four of them, named Sommerville No. 1–4 by Goldberg [Gol], tiled Euclidean space in an orientation-preserving manner. Gallagher et al. [GGH] computed that the surface-area minimizing tetrahedron among these is the Sommerville No. 1 (Figure 1), which has four congruent isosceles sides with two edges of length $\sqrt{3}$ and one edge of length 2. It can be obtained by slicing a triangular prism into three congruent tetrahedra (Figure 1).

Edmonds pointed out that the Sommerville (xi) tetrahedron (with $d_{12} = d_{23} = d_{34} = p$ and $d_{13} = d_{14} = d_{24} = q$) was neglected in Sommerville's original paper, but proved that it sometimes tiles face-to-face, although in a non-orientation-preserving manner. We present this result as Proposition 3.1, which is used several times throughout our paper.

Proposition 3.1. The Sommerville (xi) tetrahedron (with $d_{12} = d_{23} = d_{34} = p$ and $d_{13} = d_{14} = d_{24} = q$) tiles face-to-face precisely when $p/q = \sqrt{2/3}$ or $\sqrt{3/2}$. However, it tiles only in a non-orientation-preserving manner.

Proof. Proven by Edmonds [Ed].

Hill and Baumgartner. Hill and Baumgartner [Hi, Ba1, Ba2] both contributed to early work in tetrahedral tilings. We generally do not refer to their tetrahedra by name, since all except Baumgartner's T2 (Hill's second type) are subcases of Sommerville No. 1–4. Baumgartner's T2 is a subcase of Goldberg's second and third infinite families

(Prop. 3.5). A summary of Hill's, Baumgartner's, and Sommerville's work is provided by Goldberg [Gol].

Goldberg. Following papers of Sommerville and Baumgartner, Goldberg [Gol] presented three infinite families of tetrahedral tiles (Figs. 2–4), which in general need reflections to tile. Sommerville No. 1 and No. 3 appear as degenerate cases of tiles which are their own reflections.

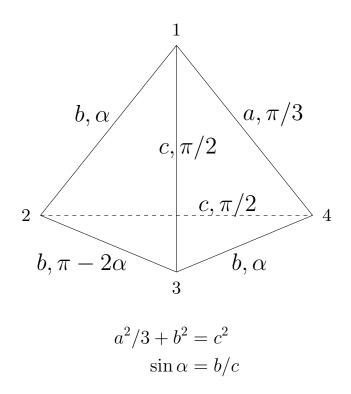


FIGURE 2. Goldberg's first family labeled with edge lengths and dihedral angles.

Goldberg also considered non-face-to-face tiles. In fact, Goldberg's first and second families tile face-to-face while Goldberg's third family tiles only non-face-to-face in general.

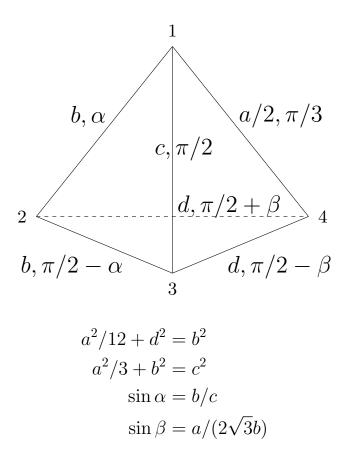


FIGURE 3. Goldberg's second family labeled with edge lengths and dihedral angles.

Proposition 3.2. Goldberg's first and second tiling families can be face-to-face but not necessarily orientation-preserving.

Proof. Consider Goldberg's first family. Goldberg derives it by dissecting an equilateral triangular prism into tetrahedra that are congruent without reflection as in Figure 5. The tetrahedra may be stacked into an arbitrarily long prism. Each face of the prism is composed of triangles with side lengths 3a, b and c, and all faces are congruent without reflection, as illustrated in Figure 6.

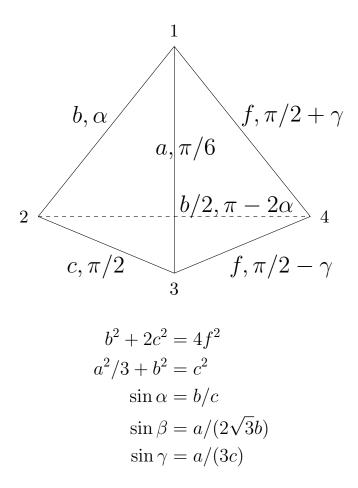


FIGURE 4. Goldberg's third family labeled with edge lengths and dihedral angles.

Since the prism has an equilateral triangular cross section, it can tile space. In order for the tetrahedra within to tile face-to-face, a reflection of the prism must also be used (except for cases where the tetrahedron does not have a distinct reflection), as illustrated in Figure 6. Then the prism and its reflection can be arranged as in Figure 6b to tile space, and the tetrahedra within tile face-to-face.

Goldberg's second family is derived from slicing the tetrahedra in Goldberg's first family in half, as in Figure 7. To show that it can tile

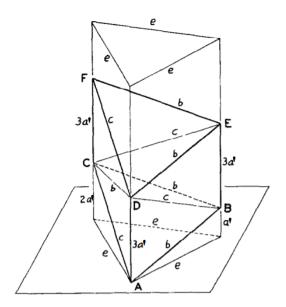


FIGURE 5. Derivation of Goldberg's first family [Gol]. We have primed a in Goldberg's image; in our paper, a = 3a'.

face-to-face, notice that this slicing cuts the triangles on the face of the prism in half in such a way that the prism and its reflection can still match up face-to-face as in Figure 6.

Remark 3.3. Note that by reflecting the prism as in Figure 6, the tetrahedra within are also reflected, and the tile is no longer orientation-preserving in general.

Proposition 3.4. Goldberg's third family does not tile face-to-face in general.

Proof. In general, a tetrahedron in Goldberg's third family has only one edge of length b which has dihedral angle α which can vary continuously. In a face-to-face tile, the tetrahedra must stack around the edge of length b, and the dihedral angles around that edge must sum

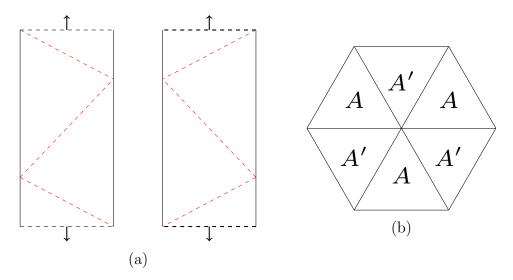


FIGURE 6. (6a): The rectangular section of the face of a prism sliced into tetrahedra that are congruent without reflection (left). In order for the tetrahedra within to tile face-to-face, a reflection of the prism (right) must be used. (6b): Tiling of Goldberg prisms. A denotes the original prism, and A' is the mirror image. The prism and its reflection tile space face-to-face, proving that Goldberg's first family tiles face-to-face.

to 2π . This means that some multiple of α must be 2π , so α cannot vary continuously.

We now show that the least-area tetrahedral tile among previously identified tiles is the Sommerville No. 1. We start by showing that the Goldberg families include many of the previous tiles.

Proposition 3.5. All previously identified tetrahedral tiles except Sommerville No. 2 and Sommerville No. 4 are included in Goldberg's families.

Proof. We consider each previously mentioned tetrahedron. Many of these justifications are also made by Goldberg [Gol].

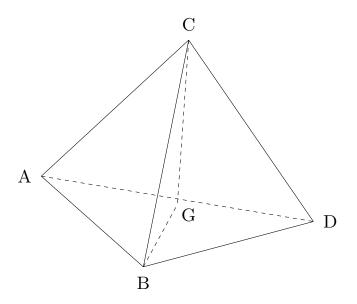


FIGURE 7. Goldberg's second family is derived by slicing a tetrahedron ABCD of the first family into congruent halves ACBG and BCDG. Vertex labeling corresponds to Goldberg's own labeling (Figure 5).

- (1) Sommerville No. 1: First Goldberg family with $\alpha = \pi/3$.
- (2) Sommerville No. 2: Not in the Goldberg families.
- (3) Sommerville No. 3: Second Goldberg family with $\alpha = \pi/4$.
- (4) Sommerville No. 4: Not in the Goldberg families.
- (5) Sommerville (xi), $p/q = \sqrt{2/3}$: First Goldberg family with 3a = c.
- (6) Baumgartner T2: Second Goldberg family with $\alpha = \pi/3$. Also third Goldberg family with $\alpha = \pi/3$.

Now we identify the least-area tetrahedron among Goldberg's infinite families.

Theorem 3.6. The least-area tetrahedron among Goldberg's infinite families is the Sommerville No. 1.

Proof. Let V be the volume of a tetrahedron and S be its surface area. To minimize the surface area given unit volume, we minimize the quantity S^3/V^2 , which is invariant up to scaling. Given edge lengths, the volume can be written in terms of the Cayley-Menger determinant as in Lemma 2.5. The surface area is the sum of the face areas calculated using Heron's formula:

$$S = \sqrt{s_1(s_1 - d_{23})(s_1 - d_{24})(s_1 - d_{34})} + \sqrt{s_2(s_2 - d_{13})(s_2 - d_{14})(s_2 - d_{34})} + \sqrt{s_3(s_3 - d_{12})(s_3 - d_{14})(s_3 - d_{24})} + \sqrt{s_4(s_4 - d_{12})(s_4 - d_{13})(s_4 - d_{23})},$$

where

$$s_i = \frac{d_{jk} + d_{kl} + d_{jl}}{2}$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Each of Goldberg's families have one degree of freedom and therefore S^3/V^2 can be written as a function of one parameter. We then find the minimum of this function with Mathematica.

(1) First family: $d_{12} = 1$, $d_{13} = \sqrt{3a^2 + 1}$, $d_{14} = 3a$, $d_{23} = 1$, $d_{24} = \sqrt{3a^2 + 1}$, $d_{34} = 1$.

The minimum is ~ 7.413 at a = 1/3.

- (2) Second family: $d_{12} = 1$, $d_{13} = \sqrt{3a^2 + 1}$, $d_{14} = 3a/2$, $d_{23} = 1$, $d_{24} = \sqrt{4 3a^2}/2$, $d_{34} = \sqrt{4 3a^2}/2$. The minimum is ~ 8.109 at $a \approx 0.491$.
- (3) Third family: $d_{12} = 1$, $d_{13} = \sqrt{6a^2 + 3}/2$, $d_{14} = 3a$, $d_{23} = 1/2$, $d_{24} = \sqrt{3a^2 + 1}$, $d_{34} = \sqrt{6a^2 + 3}/2$.

The minimum is ~ 8.273 at $a \approx 0.238$.

From these calculations, Goldberg's first family with a = 1/3 is the least-area tetrahedron, and this tetrahedron is the Sommerville No. 1. The code for this computation is in the GitHub repository in the file GoldbergTetrahedra.nb.

From Proposition 3.5 and Theorem 3.6 we can deduce the least-area tetrahedral tile among all previously identified tiles.

Corollary 3.7. The least-area tile among all previously identified tetrahedral tiles is the Sommerville No. 1.

Proof. By Proposition 3.5, all previously identified tetrahedral tiles are Sommerville No. 2 and No. 4 and the Goldberg infinite families. Gallagher et al. [GGH] calculated that the Sommerville No. 1 is the least-area tetrahedron among all the Sommerville cases, and by Theorem 3.6 the Sommerville No. 1 is the least-area tetrahedron among the Goldberg infinite families.

4. Edge-Length Graphs

In Section 4 we define a combinatorial object called an edge-length graph, which is used to show that certain linear combinations of dihedral angles sum to 2π (Prop. 4.3). We continue to assume that tilings are face-to-face. Here we consider only tetrahedra, but this concept can be easily generalized to other polyhedra.

Definition 4.1. Let T be a tetrahedron and d one of its edge lengths. We define the d-edge-length graph to be the graph where for $\{i, j, k, l\} =$ $\{1,2,3,4\}$, if edge e_{ij} has length d, the ordered pairs of edge lengths (d_{ik}, d_{jk}) and (d_{il}, d_{jl}) are an edge of the graph. We label this graph edge with the unordered pair of numbers $\{i, j\}$.

Notice that if $(d_{ik}, d_{jk}) = (d_{il}, d_{jl})$, then this graph edge is a loop. Each graph edge can have multiple labels.

The meaning of the edge-length graph is given in Lemma 4.2, which states that going from node to node in the edge-length graph is equivalent to stacking a tetrahedron around an edge. This concept is illustrated in Figure 8.

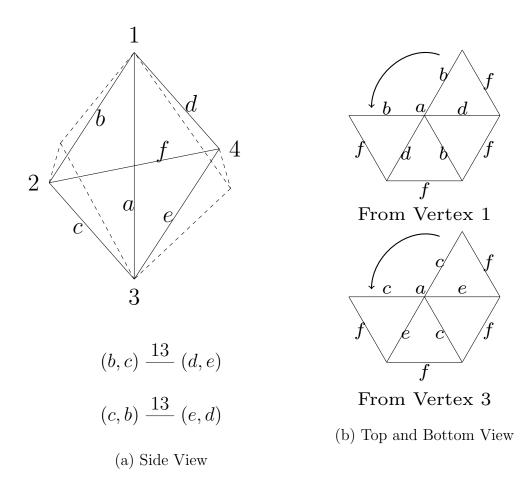


FIGURE 8. The a-edge-length graph for a tetrahedron with edge lengths a, b, c, d, e, f all distinct. The first line describes that the edge with length a, e_{13} , connects (b, c) to (d, e). The second line describes that the edge with length a, e_{13} , connects (c, b) to (e, d). From this graph and Proposition 4.3, $n\theta_{13} = 2\pi$ for some $n \in \mathbb{N}$.

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Lemma 4.2. Let T be a tetrahedron. Suppose you have disjoint copies T_i of T stacked around a line segment PQ (with one edge of each copy coinciding with PQ, so that the sum of the dihedral angles is 2π). Copy T_i of T has faces PQR_i and PQR_{i+1} . Then (PR_i, QR_i) and (PR_{i+1}, QR_{i+1}) are an edge in the d-edge-length graph of T. In other words.

$$(PR_1, QR_1), \ldots, (PR_n, QR_n), (PR_1, QR_1)$$

is a closed walk in the d-edge-length graph of T.

Proof. This follows directly from the definition of the edge-length graph.

The following proposition provides information on linear combinations of dihedral angles from the edge-length graph.

Proposition 4.3. Let T be a tetrahedral tile and d one of its edge lengths. From any node in the d-edge-length graph of T, there are a closed walk of length n starting and ending at that node and labels $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}$, where $\{i_k, j_k\}$ is a label of the kth edge of the closed walk, such that

$$\theta_{i_1j_1} + \dots + \theta_{i_nj_n} = 2\pi.$$

Proof. Any node of the edge-length graph corresponds to at least one pair of adjacent edges of a face of T. Fix a tiling, and consider how T stacks around the third edge of the face. By Lemma 4.2, there are a closed walk starting and ending at that node and labels $\{i_k, j_k\}$ associated with the kth edge of the closed walk such that the angles $\theta_{i_k j_k}$ are the dihedral angles between copies of T in the stack, summing to 2π .

Corollary 4.4. Given an edge length d, for all edges e_{ij} of length d, there are nonnegative integers n_{ij} such that

$$\sum n_{ij}\theta_{ij}=2\pi.$$

That is, some linear combination of the dihedral angles associated with edges of length d sums to 2π . Moreover, there is a closed walk in the d-edge-length graph such that the label $\{i,j\}$ is passed through exactly n_{ij} times, where for a graph edge with multiple labels, we can pick any label to represent that edge each time we pass through the edge.

The following corollary is a special case of Corollary 4.4.

Corollary 4.5. Let T be a tetrahedron which tiles space face-to-face and d one of its edge lengths. If all edges with length d have the same associated dihedral angle $\theta(d)$, then $\theta(d)$ is $2\pi/n$ for some $n \in \mathbb{N}$.

Moreover, if the d-edge-length graph of T has no closed walk of odd length, then $\theta(d)$ is π/n for some $n \in \mathbb{N}$.

Proof. By Corollary 4.4,

$$\left(\sum n_{ij}\right)\theta(d) = 2\pi,$$

so $\theta(d)$ is $2\pi/n$ for some $n \in \mathbb{N}$. If the d-edge-length graph has no closed walk of odd length, then $\sum n_{ij}$ is even, so $\theta(d) = \pi/n$ for some $n \in \mathbb{N}$.

Remark 4.6. We can characterize closed walks in specific graphs. In a tree, a closed walk passes through each edge an even number of times. If a graph includes a 3-cycle whose removal leaves the three vertices pairwise disconnected, then the numbers of times the edges of this 3-cycle are traversed by a closed walk are either all even or all odd.

5. Tetrahedra with Dihedral Angles $2\pi/n$

In Section 5 Table 1, we identify all tetrahedra with dihedral angles all of the form $2\pi/n$ in Table 1. (We continue to assume that all tilings are face-to-face.) By Proposition 2.7, the (ordered) dihedral angles determine the tetrahedron up to translation, rotation, and scaling. (Reflection alters the order of the dihedral angles.) We prove that these lists are exhaustive in Theorems 5.5 and 5.6.

TABLE 1. All tetrahedra with dihedral angles of the form $\theta_{ij} = 2\pi/n_{ij}$, up to permutation of vertices (Thm. 5.5). Area denotes the surface area for a tetrahedron of volume 1. The regular tetrahedron has area about 7.21. The tiles were previously identified by Sommerville [So2] and Goldberg [Gol] (Sect. 3). The last six do not tile (Thm. 5.6).

Tiles							
	n_{12}	n_{13}	n_{14}	n_{23}	n_{24}	n_{34}	Area
Sommerville No. 3	3	6	6	8	8	4	8.18
Sommerville No. 2	4	4	4	6	6	8	7.96
First Goldberg family, $\alpha = 2\pi/8$	4	4	8	8	4	6	7.97
First Goldberg family, $\alpha = 2\pi/5$	4	5	6	10	5	4	7.90
Sommerville No. 1	4	6	6	6	6	4	7.41
Non	-tiles	3					
	n_{12}	n_{13}	n_{14}	n_{23}	n_{24}	n_{34}	Area
NT(A)	3	4	5	10	6	6	8.81
NT(B)	3	5	5	10	10	4	8.81
NT(C)	3	5	10	10	6	4	8.84
NT(D)	3	6	10	10	10	3	9.28
NT(E)	4	4	4	5	6	10	8.64
NT(F)	4	5	6	5	6	5	7.53

First we check that the dihedral angles of tetrahedra in Table 1 form valid tetrahedra. This is not trivial since there are no simple sufficient conditions on the dihedral angles.

Lemma 5.1. In any tetrahedron, for any $\{i, j, k, l\} = \{1, 2, 3, 4\}$,

$$d_{kl}: d_{jl}: d_{jk} = |F_j| \sin \theta_{kl}: |F_k| \sin \theta_{jl}: |F_l| \sin \theta_{jk}.$$

Proof. If h_i is the height corresponding to face F_i , then by projection

$$|F_j|\sin\theta_{kl} = \frac{1}{2}h_i d_{kl}.$$

Thus the ratio in the lemma statement holds by canceling out h_i .

For some proofs we will need to compute the edge lengths of a tetrahedron given its dihedral angles. This computation is described in Lemma 5.2.

Lemma 5.2. Given the dihedral angles of a tetrahedron, the edge lengths can be computed (up to scaling).

Proof. Lemma 2.3 can be used to compute an area vector of the tetrahedron, up to scaling. By Lemma 5.1, for any $\{i, j, k, l\} = \{1, 2, 3, 4\}$ the ratio

$$d_{kl}:d_{jl}:d_{jk}$$

can be computed. From this and the areas of the faces, all edge lengths can be computed using simple properties of triangles. \Box

Proposition 5.3. The dihedral angles specified in Table 1 form valid tetrahedra.

Proof. We check that the determinant condition in Lemma 2.3 is satisfied symbolically using Mathematica. This implies that the dihedral angles are exact.

To check that the dihedral angles form a valid tetrahedron, we compute edge lengths as described in Lemma 5.2 and use Lemma 2.5 to verify that these edge lengths form a tetrahedron. Then Lemma 2.6 allows us to compute the dihedral angles from edge lengths to check that we get the original dihedral angles. The last step is needed since invalid dihedral angles may give rise to valid edge lengths. The code for this computation is in the GitHub repository in the folder 2pi_over_n.

All tetrahedral tiles in Table 1 have been previously confirmed as tiles by Goldberg [Gol] (Sect. 3). The Sommerville tiles have the same name as in the paper by Goldberg, and the First Goldberg family refers to the first of the three Goldberg infinite families. Recall that by Proposition 3.2, the First Goldberg family tiles face-to-face.

We now prove that Table 1 is exhaustive. The approach is to first show that each dihedral angle cannot be very small, and so the problem reduces to a finite search. A computer program was written to perform the search, yielding eleven candidate tetrahedra in Table 1. Finally, we show that exactly five of them tile space (face-to-face), as indicated in Table 1.

Lemma 5.4. If all dihedral angles of a tetrahedron are of the form $\theta_{ij} = 2\pi/n_{ij}$, then every n_{ij} is less than 42.

Proof. Suppose to the contrary that $n_{12} \ge 42$, so that $\theta_{12} \le 2\pi/42$. We will arrive at a contradiction using inequalities on the dihedral angles.

By Lemma 2.4, $\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}<2\pi$. Therefore either $\theta_{13}+\theta_{14}<\pi$ or $\theta_{23}+\theta_{24}<\pi$. Assume that $\theta_{13}+\theta_{14}<\pi$. The other inequality in Lemma 2.4 implies

$$\frac{2\pi}{42} + \theta_{13} + \theta_{14} \ge \theta_{12} + \theta_{13} + \theta_{14} > \pi,$$

so that $\theta_{13} + \theta_{14} > 20\pi/21$. Hence

$$\frac{20\pi}{21} < \theta_{13} + \theta_{14} < \pi.$$

We will show that this last inequality is impossible.

Notice that one of θ_{13} and θ_{14} has to be greater than $10\pi/21$, which is greater than $2\pi/5$. Assume that $\theta_{13} > 2\pi/5$. So θ_{13} is either $2\pi/3$ or $2\pi/4$. If $\theta_{13} = 2\pi/3$, then (1) reduces to

$$\frac{2\pi}{7} < \theta_{14} < \frac{2\pi}{6},$$

which is a contradiction. On the other hand, if $\theta_{13} = 2\pi/4$, then (1) reduces to

$$\frac{2\pi}{5} < \frac{19\pi}{42} < \theta_{14} < \frac{2\pi}{4},$$

which is also a contradiction. We conclude that it is impossible to have $n_{12} \geq 42$.

Theorem 5.5. The only tetrahedra with dihedral angles of the form $2\pi/n$, where $n \in \mathbb{N}$, are those identified in Table 1, up to permutation of vertices (reflection included).

Proof. By Lemma 5.4, all dihedral angles are of the form $2\pi/n$, where $3 \le n \le 41$ is a natural number. A computer search can be performed to find all sextuples $(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34})$ of numbers of this form which satisfy the determinant condition in Lemma 2.3. Moreover, the inequality conditions in Lemma 2.4 are used to narrow down many superfluous possibilities. Finally, permutation of vertices is taken into account to reduce the cut down search time and eliminate redundant results. The final results are those in Table 1. The code for the computation is in the file $2pi_over_n/search.m$.

Theorem 5.6. The only face-to-face tetrahedral tiles with dihedral angles of the form $2\pi/n$, where $n \in \mathbb{N}$, are those indicated in Table 1, up to permutation of vertices (reflection included).

Proof. By Theorem 5.5, we only need to show that the six so-called non-tiling tetrahedra in Table 1 indeed do not tile face-to-face. The main tool used is Corollary 4.5.

From the dihedral angles of Table 1, we can compute the edge lengths using the method outlined in Lemma 5.2. Note that this computation is numerical, so we can only confirm that edge lengths are distinct and not that they are identical. Therefore, we must consider separately the case where all edges that we claim have the same length do indeed have the same length and the case where they in fact do not have the same length.

Suppose first that all edges that we claim to have the same length do indeed have the same length. The proof for each of the non-tiling tetrahedra in Table 1 is given in Table 2. For each tetrahedron, we identify an edge e_{ij} of length d for which all edges of length d have the same associated dihedral angle of the form $2\pi/n$ for n odd and the d-edge-length graph has no closed walk of odd length. However,

Table 2. Edge-length graphs of all non-tiles identified in Table 1. All the graphs have no closed walk of odd length, so by Corollary 4.5, θ_{ij} should be π/n for some n, which is not the case; hence these tetrahedra cannot tile.

Name	Symmetries	Angle θ_{ij}	d-Edge-Length Graph
NT(A)	None	$\theta_{12} = 2\pi/3$	$(d_{13}, d_{23}) \xrightarrow{12} (d_{14}, d_{24})$
NT(B)	$d_{13} = d_{14} d_{23} = d_{24}$	$\theta_{13} = \theta_{14} = 2\pi/5$	$(d_{12}, d_{23}) \xrightarrow{13=14} (d_{14}, d_{34})$
NT(C)	$d_{13} = d_{14}$	$\theta_{12} = 2\pi/3$	$(d_{13}, d_{23}) \xrightarrow{12} (d_{14}, d_{24})$
NT(D)	$d_{12} = d_{34} d_{14} = d_{23}$	$\theta_{12} = \theta_{34} = 2\pi/3$	$(d_{13}, d_{23}) \xrightarrow{12=34} (d_{14}, d_{24})$
NT(E)	None	$\theta_{23} = 2\pi/5$	$(d_{12}, d_{13}) \xrightarrow{23} (d_{24}, d_{34})$
NT(F)	$d_{13} = d_{23} d_{14} = d_{24}$	$\theta_{13} = \theta_{23} = 2\pi/5$	$(d_{12}, d_{23}) \frac{13=23}{(d_{14}, d_{34})}$

by Corollary 4.5, if the tetrahedron tiled then the associated angle θ_{ij} would be of the form π/n . Therefore, these tetrahedra do not tile.

Now, suppose that some edges that we claim to have the same length in fact have distinct lengths. We can easily check that even if some edges that we claim to have the same length in fact have distinct lengths, the edge-length graphs will still have no closed walk of odd length. Then by Corollary 4.5, if the tetrahedron tiled then the associated angle θ_{ij} would be of the form π/n , so these tetrahedra do not tile.

6. Classification of Tetrahedra

In Section 6, we classify all tetrahedra by distinct edge lengths. The tetrahedra are classified into 25 types, up to permutation of vertices, as listed explicitly in Table 3. Different letters signify different edge lengths. The summary of these 25 types, categorized according to the number of distinct edge lengths and the number of congruent faces and sorted into several groups, is given in Figure 9.

After writing up this classification into 25 types and sending a copy to Wirth, he kindly sent us his 2013 version [WD1].

First, it is shown that all 25 types exist.

Proposition 6.1. All 25 types of tetrahedra presented in Table 3 occur.

Proof. By Lemma 2.5, any sextuple of edge lengths satisfying certain inequalities forms a tetrahedron. Then the edge lengths of a regular tetrahedron can be adjusted slightly to make the tetrahedron belong to any of the 25 types, and the edge lengths will still satisfy the inequalities. \Box

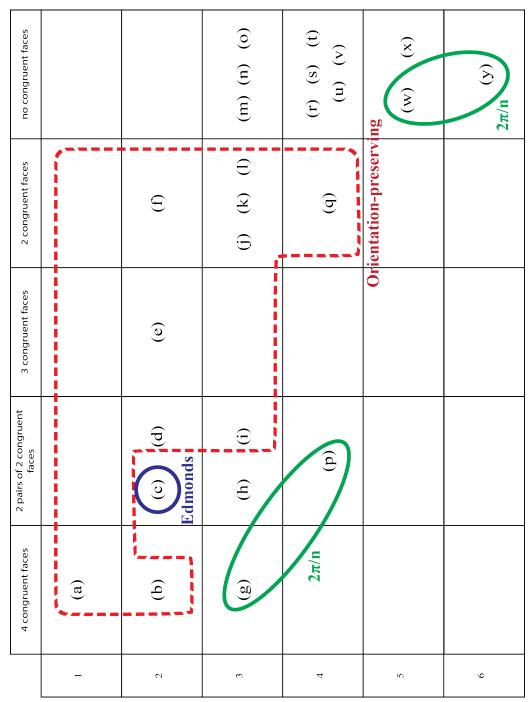
Proposition 6.2. The 25 types of tetrahedra presented in Table 3 are complete.

Proof. Our proof is the same as Wirth and Dreiding's [WD1, Sect. 1]. We consider the number of distinct edge lengths and how many times each length occurs, which Wirth and Dreiding call the *lengths partition*. For each lengths partition, several arrangements of edge lengths into tetrahedra can be found; it is easy to check that these arrangements are the only ones possible and are unique up to rotation, translation, scaling, and reflection, resulting in the complete list of 25 total types of tetrahedra presented in Table 3.

Proposition 6.3. The Sommerville Nos. 1–4 are the only tiles of the ten types in the orientation-preserving group of Figure 9.

Proof. It is easy to check that the ten types of the orientation-preserving group are their own reflections. Therefore by Definition 2.1 these ten

types can only tile in an orientation-preserving manner. An incomplete proof by Sommerville [So1], later completed by Edmonds [Ed], shows that the Sommerville No. 1–4 are the only orientation-preserving tiles. $\hfill\Box$



Number of distinct edge lengths

FIGURE 9. Classification of all tetrahedra into 25 types, sorted by number of distinct edge lengths and number of congruent faces. The only tiles in the *orientation-preserving group* are the Sommerville Nos. 1–4 (Prop. 6.3). The $Edmonds\ group$ contains one tile (Prop. 3.1). The $2\pi/n\ group$ contains no tiles (Thm. 6.6). The rest, referred to as non-characterized, may have incomplete lists of tiles and are considered in Sections 7–10.

Table 3. Classification of tetrahedra into 25 types.

Type	Picture	Group	Tiles
(a)	$ \begin{array}{c c} 1 \\ a \\ a \\ a \end{array} $ $ \begin{array}{c} a \\ a \end{array} $ $ \begin{array}{c} a \\ a \end{array} $	Orientation- preserving	Does not tile.
(b)	$ \begin{array}{c} 1 \\ a \\ b \\ a \end{array} $ $ \begin{array}{c} a \\ b \\ a \end{array} $	Orientation- preserving	Tiles only if Sommerville No. 1 $(a/b = \sqrt{3}/2)$.
(c)	$ \begin{array}{c c} 1 \\ a \\ b \\ 3 \end{array} $	Edmonds	Tiles only if $a/b = \sqrt{2/3}$ or $\sqrt{3/2}$ by Edmonds [Ed].
(d)	$ \begin{array}{c} 1 \\ a \\ a \\ a \end{array} $ $ \begin{array}{c} a \\ b \end{array} $ $ \begin{array}{c} 4 \end{array} $	Orientation- preserving	Does not tile.

Type	Picture	Group	Tiles
(e)	$ \begin{array}{c} 1 \\ a \\ a \\ b \\ 3 \end{array} $	Orientation- preserving	Does not tile.
(f)	$ \begin{array}{c} 1 \\ b \\ a \\ a \end{array} $ $ \begin{array}{c} a \\ a \end{array} $ $ \begin{array}{c} 4 \end{array} $	Orientation- preserving	Does not tile.
(g)	$ \begin{array}{c c} 1 \\ a \\ b \\ c \\ 3 \end{array} $	$2\pi/n$	Does not tile.
(h)	$ \begin{array}{c c} 1 \\ 2 & b \\ c & b \end{array} $	Non-characterized	Tiles if in first Goldberg family $(c^2 = b^2 + a^2/3)$.

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Type	Picture	Group	Tiles
(i)	$ \begin{array}{c} 1 \\ a \\ b \\ c \\ a \end{array} $ $ \begin{array}{c} a \\ 4 \end{array} $	Orientation- preserving	Does not tile.
(j)	$ \begin{array}{c} 1 \\ a \\ a \\ c \\ b \\ 3 \end{array} $	Orientation- preserving	Tiles only if Sommerville No. 3 $(a:b:c=\sqrt{3}:2:2\sqrt{2})$ or No. 4 $(a:b:c=\sqrt{5}/2:\sqrt{3}:2)$.
(k)	$ \begin{array}{c c} 1 \\ b \\ c \\ a \\ 3 \end{array} $	Orientation- preserving	Does not tile.
(1)	$ \begin{array}{c c} 1 \\ a \\ b \\ c \\ 3 \end{array} $	Orientation- preserving	Does not tile.

Type	Picture	Group	Tiles
(m)	$ \begin{array}{c} 1 \\ 2 \\ $	Non-characterized	
(n)	$ \begin{array}{c c} 1 & a \\ $	Non-characterized	
(o)	$ \begin{array}{c} 1 \\ a \\ b \\ c \\ 3 \end{array} $	Non-characterized	
(p)	$ \begin{array}{c} 1 \\ a \\ c \\ d \\ a \end{array} $ $ \begin{array}{c} 4 \\ \end{array} $	$2\pi/n$	Does not tile.

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Type	Picture	Group	Tiles
(q)	$ \begin{array}{c c} 1 \\ a \\ c \\ d \\ d \\ d \\ 4 \end{array} $	Orientation- preserving	Tiles only if Sommerville No. 2 $(a:b:c:d=\sqrt{3}:\sqrt{2}:2:1)$.
(r)	$ \begin{array}{c} 1 \\ a \\ a \\ c \\ d \end{array} $ $ \begin{array}{c} a \\ c \\ d \end{array} $	Non-characterized	
(s)	$ \begin{array}{c} 1 \\ a \\ b \\ d \\ d \end{array} $ $ \begin{array}{c} d \\ 4 \end{array} $	Non-characterized	
(t)	$ \begin{array}{c} 1 \\ a \\ b \\ c \\ d \end{array} $	Non-characterized	

Type	Picture	Group	Tiles
(u)	$ \begin{array}{c} 1 \\ a \\ a \\ b \\ 3 \end{array} $	Non-characterized	
(v)	$ \begin{array}{c c} 1 \\ a \\ b \\ c \\ 4 \end{array} $	Non-characterized	Tiles if in second Goldberg family $(a^2 = c^2 + d^2/3, b^2 = a^2 + 4d^2/3)$, including Baumgartner T2 $(a:b:c:d=\sqrt{3}:2:\sqrt{11/2}:\sqrt{3/2})$.
(w)	$ \begin{array}{c} 1 \\ b \\ a \\ a \\ e\end{array} $ $ \begin{array}{c} 4 \\ \end{array} $	$2\pi/n$	Does not tile.
(x)	$ \begin{array}{c c} 1 \\ a \\ b \\ d \\ e \\ 4 \end{array} $	Non-characterized	

0	- 4
• 5	4

Type	Picture	Group	Tiles
(y)	$ \begin{array}{c c} 1 \\ a \\ b \\ c \\ e \\ 3 \\ f \end{array} $	$2\pi/n$	Does not tile.

We now show that tetrahedra of the $2\pi/n$ group—types (g), (p), (w), and (y)—do not tile (face-to-face). We first need the following lemmas describing the symmetries of tetrahedra of types (g) and (p).

Lemma 6.4. For a tetrahedron of type (g), the opposite dihedral angles are equal: $\theta_{12} = \theta_{34}$, $\theta_{13} = \theta_{24}$, and $\theta_{14} = \theta_{23}$.

Proof. Permute the vertices so that vertices V_2 and V_3 are switched and vertices V_1 and V_4 are switched. The resulting tetrahedron is identical to the original. Hence $\theta_{12} = \theta_{34}$. The other equalities follow from analogous reasoning.

Lemma 6.5. For a tetrahedron of type (p), $\theta_{12} = \theta_{34}$ and $\theta_{14} = \theta_{23}$.

Proof. Permute the vertices so that vertices V_1 and V_3 are switched and vertices V_2 and V_4 are switched. The resulting tetrahedron is identical to the original one. Therefore $\theta_{12} = \theta_{34}$ and $\theta_{14} = \theta_{23}$.

The following theorem shows that tetrahedra in the $2\pi/n$ group, types (g), (p), (w), and (y), do not tile.

Theorem 6.6. A tetrahedron of type (g), (p), (w) or (y) does not tile (face-to-face).

Proof. First, we show that all dihedral angles of a tetrahedral tile of these types must be of the form $2\pi/n$. For types (g), (p), and (w), the edge-length graph is used to prove this result.

TABLE 4. Edge-length graphs of tetrahedra of types (g), (p), and (w). For edge lengths that do not appear in this table, there is only one edge of that length, and the corresponding edge-length graph consists of two nodes connected by an edge.

Type	Edge-Length Graph(s)
	For edge length a : $(b,c) \frac{12=34}{} (c,b)$
(g)	For edge length b : $(a,c) \frac{13=24}{} (c,a)$
	For edge length c : $(a,b) \frac{14=23}{} (b,a)$
	For edge length a :
	$(b,c) \frac{12=34}{} (d,b)$
(p)	$(b,d) \frac{12=34}{} (c,b)$
	For edge length b :
	$(a,c) \frac{14=23}{(d,a)}$
	$(a,d) \frac{14=23}{2} (c,a)$

Type	Edge-Length Graph(s)
(w)	For edge length a : $(b,d) \frac{13}{} (c,e)$ $(d,b) \frac{13}{} (e,c)$ $(b,c) \frac{24}{} (d,e)$ $(c,b) \frac{24}{} (e,d)$

- Type (g): By Lemma 6.4, $\theta_{12} = \theta_{34}$, $\theta_{13} = \theta_{24}$, and $\theta_{14} = \theta_{23}$. By Corollary 4.5, every dihedral angle is of the form $2\pi/n$.
- Type (p): By Lemma 6.5, $\theta_{12} = \theta_{34}$ and $\theta_{14} = \theta_{23}$. By Corollary 4.5, every dihedral angle is of the form $2\pi/n$.
- Type (w): From the a-edge-length graph, any closed walk in the graph may only pass through one of the labels 13 or 24, but not both. That is, the two edges of common length cannot coincide in a tiling. Then the all dihedral angles of the tetrahedron must be of the form $2\pi/n$.
- Type (y): Every edge has a distinct length, so every dihedral angle is of the form $2\pi/n$ by Corollary 4.5.

By Theorem 5.6, a tetrahedral tile of these four types must be in Table 1. By inspection, tetrahedral tiles in Table 1 are not of these four types; hence, a tetrahedron of any of these four types does not tile.

Section 6 is summarized by the following theorem, which states that the Sommerville No. 1 minimizes surface area among the tiles of all completely characterized types. **Theorem 6.7.** The Sommerville No. 1 uniquely minimizes surface area among all tiles of the orientation-preserving, Edmonds, and $2\pi/n$ groups of Figure 9.

Proof. By Propositions 3.1 and 6.3 and Theorem 6.6, the tiles of the orientation-preserving, Edmonds, and $2\pi/n$ groups of Figure 9 are among those identified in previous literature. Then by Corollary 3.7, the Sommerville No. 1 uniquely minimizes surface area among the tiles of these groups.

It remains to be shown that the Sommerville No. 1 minimizes surface area among the non-characterized ten types of Figure 9, which are examined in Sections 7–10.

7. Previously Non-Characterized Ten Types

In Section 7, we deduce some necessary conditions on the ten non-characterized types defined in Section 6 in Figure 9. We will use these and the isoperimetric conditions of Section 8 to eliminate isoperimetric candidates of these ten types in Section 9.

First, we set up linear systems on dihedral angles of the 10 non-characterized types based on their edge-length graphs (Table 5).

Proposition 7.1. The 10 non-characterized types have the edge-length graphs and linear combinations of dihedral angles as indicated in Table 5.

Proof. The edge-length graphs are constructed as described in Section 4. We can deduce the linear combinations by considering closed walks in the graph using Proposition 4.3. Parities of the coefficients follow from Remark 4.6. For type (h), equalities of dihedral angles follow from the same argument as in Lemma 6.4. □

TABLE 5. Edge-length graphs for the (previously) non-characterized types, whose tiling properties are not entirely understood. For edge lengths that do not appear in this table, there is only one edge of that length, and the corresponding edge-length graph has no closed walk of odd length.

Type	Edge-Length Graph(s)	Linear Systems
(h)	For edge length b : $(c,b) \xrightarrow{34} (b,c)$ $13=24 \qquad 13=24$ $(a,c) \qquad (c,a)$ For edge length c : $(a,b) \xrightarrow{14=23} (b,b) \xrightarrow{14=23} (b,a)$	θ_{12} is of the form π/n . $n_{13}\theta_{13} + n_{34}\theta_{34} = 2\pi$. $\theta_{13} = \theta_{24}$. n_{13} and n_{34} are even. $\theta_{14} = \theta_{23}$ is of the form π/n .
(m)	For edge length a : $(b,c) \qquad (a,b) \xrightarrow{12} (a,c)$ $\begin{vmatrix} 23 & 24 & \\ (a,a) & \\ 24 & \\ 23 & 34 & \\ (c,a) \xrightarrow{12} (b,a) & (c,b) & \end{aligned}$	θ_{13} and θ_{14} are of the form π/n . $n_{12}\theta_{12} + n_{23}\theta_{23} + n_{24}\theta_{24} + n_{34}\theta_{34} = 2\pi.$ n_{12}, n_{23} , and n_{24} have the same parity. n_{34} is even.

Type	Edge-Length Graph(s)	Linear Systems
(n)	For edge length a : $(b,c) \xrightarrow{23} (b,a) \xrightarrow{12} (a,c)$ $\begin{vmatrix} 14 \\ (b,b) \\ 14 \end{vmatrix}$ $(c,b) \xrightarrow{23} (a,b) \xrightarrow{12} (c,a)$ For edge length b : $(b,a) \xrightarrow{13} (a,a) \xrightarrow{13} (a,b)$ $34 \begin{vmatrix} & & & & & & & & & & & \\ (a,c) & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & & & \\ &$	θ_{24} is of the form π/n . $n_{12}\theta_{12} + n_{14}\theta_{14} + n_{23}\theta_{23} = 2\pi$. n_{12}, n_{14} , and n_{23} are even. $n_{13}\theta_{13} + n_{34}\theta_{34} = 2\pi$. n_{13} and n_{34} are even.

Type	Edge-Length Graph(s)	Linear Systems
	For edge length a : $(b,a) \xrightarrow{12} (c,c) \xrightarrow{12} (a,b)$ $23 \mid \qquad 23$ $(b,c) \qquad (c,b)$ For edge length b :	$n_{12}\theta_{12} + n_{23}\theta_{23} = 2\pi.$ n_{12} and n_{23} are even. $n_{13}\theta_{13} + n_{34}\theta_{34} = 2\pi.$ n_{13} and n_{34} are even. $n_{14}\theta_{14} + n_{24}\theta_{24} = 2\pi.$ n_{14} and n_{24} are even.
(r)	For edge length a : (a,b) $12 / 13$ $(a,c) \frac{1}{14} (a,d)$ (b,a) $12 / 13$ $(c,a) \frac{1}{14} (d,a)$	θ_{23} , θ_{24} , and θ_{34} are of the form π/n . $n_{12}\theta_{12} + n_{13}\theta_{13} + n_{14}\theta_{14} = 2\pi$. n_{12} , n_{13} , and n_{14} have the same parity.

Type	Edge-Length Graph(s)	Linear Systems
(s)	For edge length d : (a,b) (a,c) (b,c) 23 24 34 (d,d) 34 24 23 (c,b) (c,a) (b,a)	θ_{12} , θ_{13} , and θ_{14} are of the form π/n . $n_{23}\theta_{23} + n_{24}\theta_{24} + n_{34}\theta_{34} = 2\pi.$ n_{23}, n_{24} , and n_{34} are even.
(t)	For edge length a : $(b, a) \xrightarrow{12} (a, c)$ $23 \mid \qquad 14$ $(d, c) \qquad (b, d)$ $(a, b) \xrightarrow{12} (c, a)$ $23 \mid \qquad 14$ $(c, d) \qquad (d, b)$	θ_{13} , θ_{24} , and θ_{34} are of the form π/n . $n_{12}\theta_{12} + n_{14}\theta_{14} + n_{23}\theta_{23} = 2\pi.$ n_{12}, n_{14} , and n_{23} are even.

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Type	Edge-Length Graph(s)	Linear Systems
(u)	For edge length a : $(b,c) \frac{12}{} (a,b) \frac{13}{} (b,d)$ $(c,b) \frac{12}{} (b,a) \frac{13}{} (d,b)$ For edge length b : $(a,c) \frac{14}{} (a,d)$ $(c,a) \frac{14}{} (d,a)$ $(c,d) \frac{23}{} (a,a) \frac{23}{} (d,c)$	θ_{24} and θ_{34} are of the form π/n . $n_{12}\theta_{12} + n_{13}\theta_{13} = 2\pi.$ $n_{12} \text{ and } n_{13} \text{ are even.}$ $\theta_{14} \text{ and } \theta_{23} \text{ are of the form } \pi/n.$
(v)	For edge length a : $(a,b) \xrightarrow{12} (c,c) \xrightarrow{12} (b,a)$ $23 \mid \qquad 23$ $(c,d) \qquad (d,c)$ For edge length c : $(b,d) \xrightarrow{14} (a,c) \xrightarrow{24} (a,d)$ $(d,b) \xrightarrow{14} (c,a) \xrightarrow{24} (d,a)$	θ_{13} and θ_{34} are of the form π/n . $n_{12}\theta_{12} + n_{23}\theta_{23} = 2\pi$. n_{12} and n_{23} are even. $n_{14}\theta_{14} + n_{24}\theta_{24} = 2\pi$. n_{14} and n_{24} are even.

Type	Edge-Length Graph(s)	Linear Systems
(x)	For edge length a : $(d, e) \xrightarrow{23} (a, b) \xrightarrow{12} (d, c)$ $(e, d) \xrightarrow{23} (b, a) \xrightarrow{12} (c, d)$	$\theta_{13}, \theta_{14}, \theta_{24}, \text{and} \theta_{34} \text{are of the form} \\ \pi/n.$ $n_{12}\theta_{12} + n_{23}\theta_{23} = 2\pi.$ $n_{12} \text{and} n_{23} \text{are even.}$

The vanishing determinant condition on the dihedral angles given in Lemma 2.3 can be strengthened given that some edge lengths are equal. The strengthened condition for nine of the ten previously non-characterized types is presented in Table 6. Type (u) is omitted because we will not use this condition in the code in Section 9; in the code, it is treated as a subcase of type (x).

First we show how equality of edge lengths can be used to derive relationships between face areas and dihedral angles.

Lemma 7.2. In a tetrahedron, for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, if $d_{ij} = d_{ik}$, then

$$|F_k|\sin\theta_{ij} = |F_j|\sin\theta_{ik}.$$

Proof. By Lemma 5.1,

$$d_{ij}: d_{ik} = |F_k| \sin \theta_{ij}: |F_j| \sin \theta_{ik}.$$

So the desired equality holds given that $d_{ij} = d_{ik}$.

Proposition 7.3. The 10 non-characterized types have the face area relationships as given in Table 6, and the matrix for each type has nontrivial null space.

Proof. The face area relationships can be derived by applying Lemma 7.2 to adjacent edges of equal lengths specified by the types. For type (h), the fact that $|F_1| = |F_2|$ and $|F_3| = |F_4|$ follows from direct inspection of edge lengths. Finally, because the vector $(|F_1|, |F_2|, |F_3|, |F_4|)^T$

is in the null space of the matrix in Lemma 2.3, these face area relationships imply that the matrices in the table have nontrivial null space. \Box

TABLE 6. Strengthened determinant conditions for the (previously) non-characterized types, whose tiling properties are not entirely understood. The C_i refers to column i of the matrix in Lemma 2.3. The matrices for the strengthened determinant condition all have nontrivial null space.

Type	Face Area Relationships	Strengthened Determinant Condition
(h)	$ F_1 = F_2 ,$ $ F_3 = F_4 ,$ $ F_1 \sin \theta_{34} = F_4 \sin \theta_{13}.$	$\left(\sin\theta_{13}(C_1+C_2) + \sin\theta_{34}(C_3+C_4)\right)$
(m)	$ F_1 \sin \theta_{23} = F_3 \sin \theta_{12},$ $ F_2 \sin \theta_{34} = F_3 \sin \theta_{24} =$ $ F_4 \sin \theta_{23}.$	$\begin{pmatrix} \sin \theta_{12} \sin \theta_{34} C_1 + \sin \theta_{23} \sin \theta_{24} C_2 \\ + \sin \theta_{23} \sin \theta_{34} C_3 + \sin \theta_{24} \sin \theta_{34} C_4 \end{pmatrix}$
(n)	$ F_1 \sin \theta_{23} = F_3 \sin \theta_{12},$ $ F_2 \sin \theta_{14} = F_4 \sin \theta_{12},$ $ F_1 \sin \theta_{34} = F_4 \sin \theta_{13}.$	$\begin{pmatrix} \sin \theta_{12} \sin \theta_{13} \sin \theta_{14} C_1 \\ + \sin^2 \theta_{12} \sin \theta_{34} C_2 \\ + \sin \theta_{13} \sin \theta_{14} \sin \theta_{23} C_3 \\ + \sin \theta_{12} \sin \theta_{14} \sin \theta_{34} C_4 \end{pmatrix}$
(o)	$ F_1 \sin \theta_{23} = F_3 \sin \theta_{12},$ $ F_1 \sin \theta_{34} = F_4 \sin \theta_{13},$ $ F_1 \sin \theta_{24} = F_2 \sin \theta_{14}.$	$\begin{pmatrix} \sin \theta_{12} \sin \theta_{13} \sin \theta_{14} C_1 \\ + \sin \theta_{12} \sin \theta_{13} \sin \theta_{24} C_2 \\ + \sin \theta_{23} \sin \theta_{13} \sin \theta_{14} C_3 \\ + \sin \theta_{12} \sin \theta_{34} \sin \theta_{14} C_4 \end{pmatrix}$

Type	Face Area Relationships	Strengthened Determinant Condition
(r)	$ F_2 \sin\theta_{13} = F_3 \sin\theta_{12},$ $ F_3 \sin\theta_{14} = F_4 \sin\theta_{13}.$	$\begin{pmatrix} & & \\ C_1 & \sin \theta_{12} C_2 + \sin \theta_{13} C_3 + \sin \theta_{14} C_4 \\ & & \end{pmatrix}$
(s)	$ F_2 \sin\theta_{34} = F_3 \sin\theta_{24} =$ $ F_4 \sin\theta_{23}.$	$\begin{pmatrix} & \\ \sin \theta_{23} \sin \theta_{24} C_2 \\ C_1 & + \sin \theta_{23} \sin \theta_{34} C_3 \\ & + \sin \theta_{24} \sin \theta_{34} C_4 \end{pmatrix}$
(t)	$ F_1 \sin\theta_{23} = F_3 \sin\theta_{12},$ $ F_2 \sin\theta_{14} = F_4 \sin\theta_{12}.$	$\begin{pmatrix} & \\ \sin \theta_{12} C_1 & \sin \theta_{12} C_2 \\ + \sin \theta_{23} C_3 & + \sin \theta_{14} C_4 \end{pmatrix}$
(u)	1	-
(v)	$ F_1 \sin\theta_{23} = F_3 \sin\theta_{12},$ $ F_1 \sin\theta_{24} = F_2 \sin\theta_{14}.$	$\begin{pmatrix} & \\ \sin \theta_{12} \sin \theta_{14} C_1 & \\ + \sin \theta_{12} \sin \theta_{24} C_2 & C_4 \\ + \sin \theta_{23} \sin \theta_{14} C_3 & \end{pmatrix}$
(x)	$ F_1 \sin\theta_{23} = F_3 \sin\theta_{12}.$	$ \left(\sin \theta_{12} C_1 + \sin \theta_{23} C_3 C_2 C_4 \\ \mid \mid \mid \mid \right) $

8. Isoperimetry

In Section 8, we establish bounds on the dihedral angles of a tetrahedron which can beat the Sommerville No. 1, which will be used in code in Section 9. First we need a lemma relating the volume of a tetrahedron to the areas of its faces.

Lemma 8.1. Let V be the volume of a tetrahedron. For $\{i, j, k, l\} = \{1, 2, 3, 4\}$, if we let θ_{ijk} to be the angle $V_iV_jV_k$, then

$$V^{2} = \frac{2}{9} |F_{j}| |F_{k}| |F_{l}| \sin \theta_{ij} \sin \theta_{ik} \sin \theta_{jik}.$$

Proof. Let h_i be the height of the tetrahedron from vertex V_i to face F_i . Moreover, let $h_{i,jk}$ be the height from vertex V_i to edge e_{jk} in the triangle $V_iV_jV_k$. Notice that

$$V = \frac{1}{3}h_j |F_j| = \frac{1}{3}h_k |F_k|.$$

Hence

$$V^{2} = \frac{1}{9} h_{j} h_{k} |F_{j}| |F_{k}|.$$

It remains to show that

$$h_j h_k = 2 |F_l| \sin \theta_{ij} \sin \theta_{ik} \sin \theta_{jik}.$$

By projection, we have

$$h_j = h_{j,ik} \sin \theta_{ik} = d_{ij} \sin \theta_{jik} \sin \theta_{ik}$$

and similarly

$$h_k = d_{ik} \sin \theta_{iik} \sin \theta_{ij}.$$

So it remains to prove

$$|F_l| = \frac{1}{2} d_{ij} d_{ik} \sin \theta_{jik},$$

which is true.

The following proposition gives bounds on dihedral angles of a tetrahedron in terms of its normalized surface area. **Proposition 8.2.** Suppose that a tetrahedron has volume V and surface area S. Then all its dihedral angles θ_{ij} must satisfy the inequality

$$\sin \theta_{ij} \ge \frac{243}{S^3/V^2}.$$

Proof. We can assume (i, j) = (1, 2). By Lemma 8.1, we can bound

$$V^2 \le \frac{2}{9} |F_2| |F_3| |F_4| \sin \theta_{12}.$$

Similarly, we have

$$V^2 \le \frac{2}{9} |F_1| |F_3| |F_4| \sin \theta_{12}.$$

So we may conclude

$$V^{2} \le \frac{2}{9} \left(\frac{|F_{1}| + |F_{2}|}{2} \right) |F_{3}| |F_{4}| \sin \theta_{12}.$$

The arithmetic mean-geometric mean inequality implies that

$$(|F_1| + |F_2|) |F_3| |F_4| \le \left(\frac{|F_1| + |F_2| + |F_3| + |F_4|}{3}\right)^3 = \frac{S^3}{27}.$$

Using this last inequality, we can get the desired bound. \Box

We can now apply this result to tetrahedra with less surface area than the Sommerville No. 1.

Corollary 8.3. A tetrahedron of unit volume with surface area less than or equal to the Sommerville No. 1 has all dihedral angles in the range $[\theta_0, \pi - \theta_0]$, where

$$\theta_0 = \sin^{-1}\left(\frac{27}{32\sqrt{2}}\right) \approx 36.63^\circ.$$

Proof. The Sommerville No. 1 has normalized surface area

$$\frac{S}{V^{2/3}} = 2^{11/6} \cdot 3^{2/3}.$$

The desired bound follows from Proposition 8.2 and direct computation. \Box

9. Code

In Section 9, we start by using information from Sections 7 and 8 to reduce all potential candidates to beat the Sommerville No. 1 tetrahedron to a finite number of cases. There are 2074 cases in total, and they are enumerated by computer code. Then we use the computer to eliminate all but 7 of them. Those remaining 7 cases will be handled by special arguments in Section 10.

Overview. For each edge-length graph for a (previously) non-characterized type, for the dihedral angles, Table 5 of Section 7 provides linear equations with nonnegative integer coefficients. By Corollary 8.3, for a tetrahedron to beat the Sommerville No. 1, its dihedral angles must all be greater than 36.5 degrees, so at most 9 of them can stack around a common edge. This means that the sum of the coefficients in each equation is at most 9, and so there are a finite number of possible sets of coefficients. Each set of coefficients is one case. We use the parities of coefficients provided in Table 5 and the symmetries of each type to reduce the number of cases. Finally, we make use of reductions from one type to another to assume that certain coefficients must be nonzero. This is important not only for reducing the number of cases but also because the degenerate cases where many coefficients are zero are in general more difficult to eliminate. The full details of this setup are discussed in Proposition 9.2.

Each case consists of a number of equations whose variables are the dihedral angles. Tables 5 and 6 give enough information so that in every case the number of equations is greater than the number of variables by exactly one. Specifically, we start with a determinant condition on the dihedral angles. Each time we make two edges have equal length, their associated dihedral angles are related to one another by a linear equation in Table 5, which produces one more variable. The equal-edge-lengths condition implies face-area relationships in Table 6, yielding one more equation. Thus the number of equations remains one more than the number of variables, and we expect that most of

the cases will produce no solution. This is indeed the case for all but 29 cases. To prove this rigorously, we use the interval minimization method provided by Stan Wagon [BLWW, Chapt. 4] to show that in the relevant domain of variables, all equations cannot be simultaneously satisfied. The details are given in Proposition 9.3.

For the 29 remaining cases that survive interval minimization, we attempt to use the computer algebra system of Mathematica to directly solve the system of equations. Those which produce no solution or whose solutions conflict with known conditions are discarded. The remaining 7 cases which cannot be eliminated by this method are considered by hand and finally eliminated in Section 10. The details are given in Proposition 9.4.

Implementation. The code considers 9 types of tetrahedra which roughly represent the 10 non-characterized types but are not exactly the same, because one code type can correspond to many of the original tetrahedra types and vice versa. We will use the phrase "code types" to refer to the 9 types in the code and the phrase "original types" to refer to the 10 non-characterized types from Section 6. By correspondence we mean that each case encountered in a certain code type can come from any of the corresponding original types.

Table 7 maps all such correspondences. Each of the 10 non-characterized types except (u), which is treated as a subcase of (x), has one main corresponding code type. The main code type covers most of the tetrahedra in that original type, except for possible degenerate cases, which are instead covered in the non-main corresponding code types.

We first describe the setup of each code type presented in Table 9.

Lemma 9.1. A face-to-face tetrahedral tile of type (r) has the solid angle at vertex V_1 , Ω_1 , of the form $4\pi/n$ for some $n \in \mathbb{N}$.

Proof. For a tetrahedron of type (r), vertex V_1 cannot meet with other vertices in the tiling. Hence the solid angle Ω_1 must divide 4π .

TABLE 7. The correspondences between the nine code types and ten non-characterized types of tetrahedra. "Main Corr. Type" refers to the main non-characterized type corresponding to each code type. "Other Corr. Types" refers to other non-characterized types corresponding to each code type.

Code Type	Main Corr. Type	Other Corr. Types
aaabcd	(r)	_
abaacb	(n)	_
abaacd	(t)	(n)
abcaaa	(m)	_
abcacb	(o)	_
abcacd	(v)	(o)
abcade	(x)	(u), (n), (o), (v)
abccbb	(h)	_
abcddd	(s)	_

Proposition 9.2. The nine code types with assumptions and symmetries given in Table 9 cover all potential candidates to beat or tie with the Sommerville No. 1 of the ten non-characterized types, as in Table 7.

Proof. We argue that for each non-characterized type, the corresponding code types given in Table 7 are sufficient to cover all potential candidates of that type. We first focus on the main corresponding code type. Then we consider degenerate cases which correspond to the non-main code types. Finally, we argue that the symmetries are correct.

For the original types (r), (n), (t), (m), (o), (v), (x), (h), and (s), the main corresponding code type is given in Table 7. The assumptions for these code types in Table 9, except the ones that say certain n_{ij} are nonzero, are justified as follows.

- For conditions of the form $\theta_{ij} \in \{\pi/2, \pi/3, \pi/4\}$, Table 5 says that θ_{ij} is of the form π/n in the Linear Systems column. By Corollary 8.3, n can only be 2, 3 or 4.
- That θ_{ij} , θ_{jk} , θ_{ik} cannot all be $\pi/2$ is true because otherwise the faces F_i , F_j , and F_k must be perpendicular to the face F_l , and the tetrahedron degenerates into an infinite triangular prism.
- That $\theta_{ij} + \theta_{ik} + \theta_{il} > \pi$ is true because $\Omega_i = \theta_{ij} + \theta_{ik} + \theta_{il} \pi$ is the solid angle at vertex V_i (Lemma 2.4).
- That $\theta_{ik} + \theta_{il} + \theta_{jl} + \theta_{jk} < 2\pi$ is true by Lemma 2.4.
- That the matrix of each original type given in Table 6 has non-trivial null space is argued in Proposition 7.3.
- The linear systems $\sum n_{ij}\theta_{ij} = 2\pi$ and the parities of n_{ij} are given in Table 5. By Corollary 8.3, the sum of these n_{ij} is at most 9.
- For code type aaabcd, the fact that Ω_1 divides 4π follows from the fact that this code type only corresponds to type (r) and Lemma 9.1.

Now we argue for the correctness of the conditions "at least two of the n_{ij} are nonzero," henceforth called the *nonzero condition*. By Proposition 4.3, for any linear combination

$$\sum n_{ij}\theta_{ij} = 2\pi$$

in Table 9, we can assume any coefficient n_{ij} is nonzero (but perhaps not two coefficients at once). So we can assume at least two coefficients are nonzero unless all angles θ_{ij} in the linear combination are of the form $2\pi/n$. This forms the basis of the arguments described in Table 8.

The arguments in Table 8 read as follows. For each linear system (referred to by the condition number), if the nonzero condition of that system does not hold, then certain dihedral angles must be of the form $2\pi/n$, and that possibility is covered by another code type indicated in the table. For example, for type (n), if only the nonzero condition for (3) does not hold, then θ_{12} , θ_{14} and θ_{23} are of the form $2\pi/n$. By inspection, this possibility can be covered by the code type abacde. The phrase "Reduce to the $2\pi/n$ case" means that for that possibility, all dihedral angles are of the form $2\pi/n$, so by Theorem 5.6, the tetrahedron does not beat or tie with the Sommerville No. 1. For type (u), refer to the information in Table 5. Hence all cases where the nonzero conditions fail are covered by other code types or Theorem 5.6, so that the nonzero conditions can be enforced.

Table 8. Reductions of the 10 non-characterized types to non-main corresponding code types.

Type	Main Corr. Code Type	Reductions
(r)	aaabcd	(3): Reduce to the $2\pi/n$ case.
(n)	abaacb	 (3): Reduce to code type abcade (with some permutation). (4): Reduce to code type abaacd. (3) and (4): Reduce to the 2π/n case.
(t)	abaacd	(3): Reduce to the $2\pi/n$ case.
(m)	abcaaa	(3): Reduce to the $2\pi/n$ case.
(o)	abcacb	Note the cyclic symmetry between (2) , (3) , and (4) . (3) : Reduce to code type abcade. (3) and (4) : Reduce to code type abcade. (2) , (3) , and (4) : Reduce to the $2\pi/n$ case.

Type	Main Corr. Code Type	Reductions
(v)	abcacd	(3): Reduce to code type abcade (with some permutation).(4): Reduce to code type abcade.
		(3) and (4): Reduce to the $2\pi/n$ case.
(x)	abcade	(3): Reduce to the $2\pi/n$ case.
(h)	abccbb	(4): Reduce to the $2\pi/n$ case.
(s)	abcddd	(3): Reduce to the $2\pi/n$ case.
		If n_{12} and n_{13} are nonzero, reduce to code type abcade
(u)	_	(with some permutation).
		Otherwise, reduce to the $2\pi/n$ case.

Finally, the inequalities in the Symmetries column can be enforced due to the symmetries of the assumptions of each code type, which can be easily checked. Therefore the nine code types cover all potential candidates to beat or tie with the Sommerville No. 1 of the ten non-characterized types. \Box

Table 9. Assumptions of each code type. Matrix of each type refers to the matrix in the strengthened determinant condition in Table 6.

Code Type	Assumptions	Symmetries
aaabcd	 θ₂₃, θ₂₄, θ₃₄ ∈ {π/2, π/3, π/4}. θ₂₃, θ₂₄, θ₃₄ cannot all be π/2. Matrix of type (r) has nontrivial null space (d₁₂ = d₁₃ = d₁₄). n₁₂θ₁₂ + n₁₃θ₁₃ + n₁₄θ₁₄ = 2π. n₁₂ + n₁₃ + n₁₄ ≤ 9. n₁₂, n₁₃, and n₁₄ have the same parity and at least two are nonzero. Ω₁ divides 4π. 	$n_{12} \le n_{13} \le n_{14}.$ If $n_{12} = n_{13}$, then $\theta_{24} \ge \theta_{34}$. If $n_{13} = n_{14}$, then $\theta_{23} \ge \theta_{24}$.

Code Type	Assumptions	Symmetries	
abaacb	 (1) θ₂₄ ∈ {π/2, π/3, π/4}. (2) Matrix of type (n) has nontrivial null space (d₁₂ = d₁₄ = d₂₃, d₁₃ = d₃₄). (3) n₁₂θ₁₂ + n₁₄θ₁₄ + n₂₃θ₂₃ = 2π. n₁₂ + n₁₄ + n₂₃ ≤ 9. n₁₂, n₁₄, and n₂₃ are even and at least two are nonzero. (4) n₁₃θ₁₃ + n₃₄θ₃₄ = 2π. n₁₃ + n₃₄ ≤ 9. n₁₃ and n₃₄ are even and nonzero. 		
abaacd	 θ₁₃, θ₂₄, θ₃₄ ∈ {π/2, π/3, π/4}. Matrix of type (t) has nontrivial null space (d₁₂ = d₁₄ = d₂₃). n₁₂θ₁₂ + n₁₄θ₁₄ + n₂₃θ₂₃ = 2π. n₁₂ + n₁₄ + n₂₃ ≤ 9. n₁₂, n₁₄, and n₂₃ are even and at least two are nonzero. 	$n_{14} \le n_{23}$. If $n_{14} = n_{23}$, then $\theta_{13} \ge \theta_{24}$.	

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Code Type	Assumptions	Symmetries
abcaaa	 (1) θ₁₃, θ₁₄ ∈ {π/2, π/3, π/4}. (2) Matrix of type (m) has nontrivial null space (d₁₂ = d₂₃ = d₂₄ = d₃₄). (3) n₁₂θ₁₂ + n₂₃θ₂₃ + n₂₄θ₂₄ + n₃₄θ₃₄ = 2π. n₁₂ + n₂₃ + n₂₄ + n₃₄ ≤ 9. n₁₂, n₂₃, and n₂₄ have the same parity, and n₃₄ is even. At least two of n₁₂, n₂₃, n₂₄, n₃₄ are nonzero. 	$n_{23} \le n_{24}$. If $n_{23} = n_{24}$, then $\theta_{13} \ge \theta_{14}$.

Code Type	Assumptions	Symmetries
abcacb	 (1) Matrix of type (o) has nontrivial null space (d₁₂ = d₂₃, d₁₃ = d₃₄, d₁₄ = d₂₄). (2) n₁₂θ₁₂ + n₂₃θ₂₃ = 2π. n₁₂ + n₂₃ ≤ 9. n₁₂ and n₂₃ are even and nonzero. (3) n₁₃θ₁₃ + n₃₄θ₃₄ = 2π. n₁₃ + n₃₄ ≤ 9. n₁₃ and n₃₄ are even and nonzero. (4) n₁₄θ₁₄ + n₂₄θ₂₄ = 2π. n₁₄ + n₂₄ ≤ 9. n₁₄ and n₂₄ are even and nonzero. 	$(n_{12}, n_{23}) \le (n_{13}, n_{34})$ and $(n_{12}, n_{23}) \le (n_{14}, n_{24})$ in the dictionary order.

Code Type	Assumptions	Symmetries	
abcacd	 (1) θ₁₃, θ₃₄ ∈ {π/2, π/3, π/4}. (2) Matrix of type (v) has nontrivial null space (d₁₂ = d₂₃, d₁₄ = d₂₄). (3) n₁₂θ₁₂ + n₂₃θ₂₃ = 2π. n₁₂ + n₂₃ ≤ 9. n₁₂ and n₂₃ are even and nonzero. (4) n₁₄θ₁₄ + n₂₄θ₂₄ = 2π. n₁₄ + n₂₄ ≤ 9. n₁₄ and n₂₄ are even and nonzero. 		
abcade	 θ₁₃, θ₁₄, θ₂₄, θ₃₄ ∈ {π/2, π/3, π/4}. θ₁₃, θ₁₄, θ₃₄ cannot all be π/2. θ₁₄ + θ₂₄ + θ₃₄ > π. Matrix of type (x) has nontrivial null space (d₁₂ = d₂₃). n₁₂θ₁₂ + n₂₃θ₂₃ = 2π. n₁₂ + n₂₃ ≤ 9. n₁₂ and n₂₃ are even and nonzero. 	$n_{12} \le n_{23}$. If $n_{12} = n_{23}$, then $\theta_{14} \ge \theta_{34}$.	

Code Type	Assumptions	Symmetries	
abccbb	 (1) θ₁₂, θ₁₄ = θ₂₃ ∈ {π/2, π/3, π/4}. (2) Matrix of type (h) has nontrivial null space (d₁₄ = d₂₃, d₁₃ = d₂₄ = d₃₄). (3) θ₁₃ = θ₂₄. (4) n₁₃θ₁₃ + n₃₄θ₃₄ = 2π. n₁₃ + n₃₄ ≤ 9. n₁₃ and n₃₄ are even and nonzero. 		
abcddd	 θ₁₂, θ₁₃, θ₁₄ ∈ {π/2, π/3, π/4}. θ₁₂ + θ₁₃ + θ₁₄ > π. Matrix of type (s) has nontrivial null space (d₂₃ = d₂₄ = d₃₄). n₂₃θ₂₃ + n₂₄θ₂₄ + n₃₄θ₃₄ = 2π. n₂₃ + n₂₄ + n₃₄ ≤ 9. n₂₃, n₂₄, and n₃₄ are even and at least two are nonzero. 	$n_{23} \le n_{24} \le n_{34}.$ If $n_{23} = n_{24}$, then $\theta_{13} \ge \theta_{14}$. If $n_{24} = n_{34}$, then $\theta_{12} \ge \theta_{13}$.	

We now describe how the cases are eliminated. First, from the setup, there are a total of 2074 cases. Then, using interval minimization technique, this can be reduced to 29 cases (Prop. 9.3). Finally, by solving the system of equations directly, this can be further reduced to 7 cases (Prop. 9.4).

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Proposition 9.3. From the 2074 cases of the nine code types in Table 9, all but 29 cases cannot occur.

Proof. Each choice of $\theta_{ij} \in \{\pi/2, \pi/3, \pi/4\}$ and coefficients n_{ij} in Table 9 gives one case. There are a total of 2074 cases once enumerated. For each case, we explain how to set up variables and equations. Then we explain how to determine that the system of equations has no solution.

From a linear system $\sum n_{ij}\theta_{ij} = 2\pi$ consisting of k terms, one angle θ_{ij} can be written in terms of k-1 other angles, provided that the coefficient n_{ij} is nonzero. Sometimes one coefficient is nonzero across all cases of a certain type, but sometimes different coefficients have to be chosen. Once this is done, the linear system will give rise to k-1 variables. We do this for all systems of that code type to get all variables.

The equations come from the condition that the matrix of each type has nontrivial null space in Table 9. If the matrix is $4 \times n$, there are 5-n equations, each saying that an $n \times n$ determinant vanishes. The number of equations is always greater than the number of variables by one, so the system is overdetermined and we expect that most cases will produce no solution. The choice of which determinants to use affects the number of remaining cases, which for our computation is 29. To get this number, we put in some effort to choose determinants which give few remaining cases.

To determine whether determinants d_1, d_2, \ldots, d_m can vanish simultaneously, we minimize

$$d_1^2 + d_2^2 + \dots + d_m^2$$

with respect to all variables satisfying

- (1) inequalities on sums of dihedral angles given in Lemma 2.4, and
- (2) bounds on the dihedral angles given in Corollary 8.3.

For the bounds in Corollary 8.3, we use 36.5° as a numerical lower bound of θ_0 . We do not want to get very close to the exact value to

prevent round-off errors. On the other hand, it is undesirable to use values less than or equal to 36° because this gives superfluous cases containing dihedral angles $2\pi/5$.

All files in this computation are in the folder remaining_types in the GitHub repository. We ran this computation first in the files Type_xxxxx.nb, where xxxxxx is the code type. However, this uses Mathematica's NMinimize function to minimize, which does not prove that the minimum is greater than zero. To prove this rigorously, we utilize Stan Wagon's [BLWW, Chapt. 4] interval minimization code in the file IntervalMinimize.nb. However, the interval minimization code can only minimize with respect to a hyperrectangular region. Hence we make a bounding box of the region of variables that satisfies conditions (1) and (2) above. This produces one superflouous case not present in the NMinimize approach, which is of the type abcddd. Fortunately, this case can be eliminated just by inspection. The details are in the file Type_abcddd_prove.nb.

The files that use interval minimization to rigorously eliminate all but 29 cases are Type_xxxxxx_prove.nb. The summary of this elimination process is in casesummary.txt. The work of all the code described here proves the proposition.

Proposition 9.4. From the 29 cases remaining after interval minimization of Proposition 9.3, all but 7 cases, which will be dealt with in Section 10, cannot match or improve upon the Sommerville No. 1.

Proof. From the 29 remaining cases, we attempt to solve the system of equations directly (symbolically). Then each case can be eliminated if the result is one of the following:

- (1) No solution.
- (2) Solved to a closed-form solution with dihedral angles all of the form $2\pi/n$.
- (3) For code type aaabcd, solved to a closed-form solution for which the solid angle Ω_1 does not divide 4π .

That possibility (1) can be eliminated is obvious. For possibility (2), by Theorem 5.6, no tetrahedron in this case can beat or tie with the Sommerville No. 1. For possibility (3), by Lemma 9.1, the tetrahedra in this case do not tile. A case cannot be eliminated if the result is one of the following:

- (1) Cannot solve (either to a closed-form solution or to no solution).
- (2) Timeout after 5 minutes.

The files that carry out this computation are Type_xxxxxx.nb, where xxxxxx is the code type, in the folder remaining_types. The summary of this process is in casesummary.txt. After the process has been carried out, there remain 7 cases as in Section 10.

Table 10 presents the number of cases left at each stage of the elimination process for each code type. We summarize the results of this section (Props. 9.2, 9.3, and 9.4) in the following proposition.

Proposition 9.5. From the ten non-characterized types, only seven cases as dealt with in Section 10 can potentially beat or tie with the Sommerville No. 1.

10. Remaining Cases

Section 10 examines individually the cases that remain after code is used to eliminate candidates to tile with less surface area than the Sommerville No. 1. The code used to arrive at these remaining cases is described in Section 9, and a summary of the number of remaining cases in each code type is given by Table 10. We find that out of 7 remaining cases, none tile with less surface area than the Sommerville No. 1; therefore, the Sommerville No. 1 is the least-area face-to-face tetrahedral tile even among non-orientation-preserving tiles.

Notice that each code type in Table 10 may correspond to more than one non-characterized type (Table 7). For example, for code type abaacd, the edge lengths b and d may not be different if the

TABLE 10. Summary of results from code. The main corresponding type to each code type is also given. "Num. Cases" gives the number of cases that fit initial restrictions (Prop. 9.2). "After Int. Min." gives the number of cases remaining after interval minimization (Prop. 9.3). "After Eliminate" gives the number of cases remaining after elimination based on logical restrictions (Prop. 9.4). A dash "—" in the "After Eliminate" column indicates that the type was already down to 0 cases and elimination was not applied. This process is outlined in Section 9. After coding, a total of 7 cases remain; these will be considered in Section 10.

Code Type	Main Corr. Type	Num. Cases	After Int. Min.	After Eliminate
aaabcd	(r)	224	7	2
abaacb	(n)	396	1	1
abaacd	(t)	315	8	2
abcaaa	(m)	459	0	_
abcacb	(o)	91	0	_
abcacd	(v)	324	6	1
abcade	(x)	144	5	0
abccbb	(h)	54	2	1
abcddd	(s)	67	0	_
	Total	2074	29	7

tetrahedron comes from type (n). In general, edge lengths denoted by the same letter are equal, but edge lengths denoted by different letters may or may not be equal.

The remaining cases are not particularly special; they are artifacts of the way the code was implemented (Prop. 9.3). Choosing different determinants to base the code upon may produce different cases and possibly different numbers of cases. Regardless, there will be no cases that both tile and have less surface area than the Sommerville No. 1.

Remaining case 1 (RC1). RC1 belongs to the code type aaabcd. It is characterized by

(2)
$$RC1: \begin{cases} \theta_{23} = \pi/3, \\ \theta_{24} = \pi/2, \\ \theta_{34} = \pi/4, \\ \theta_{12} + \theta_{13} + 3\theta_{14} = 2\pi. \end{cases}$$

Proposition 10.1. RC1 does not tile.

Proof. By Table 7, RC1 can only be of type (r). By Lemma 9.1, if RC1 tiles, then the solid angle

$$\Omega_1 = \theta_{12} + \theta_{13} + \theta_{14} - \pi$$

(Lemma 2.4) must divide 4π . From interval minimization, we have obtained the interval in which θ_{12} and θ_{13} must lie. By (2), we can compute the interval in which θ_{14} must lie. Then we can compute the interval containing the solid angle Ω_1 and find that

$$\frac{4\pi}{\Omega_1} \in [17.3205, 17.4558]$$
.

So Ω_1 does not divide 4π , a contradiction. The code for this computation is in the file Type_aaabcd_remaining.nb.

Remaining case 2 (RC2). RC2 belongs to the code type aaabcd. It is characterized by

(3)
$$RC2: \begin{cases} \theta_{23} = \pi/4, \\ \theta_{24} = \pi/2, \\ \theta_{34} = \pi/4, \\ \theta_{12} + \theta_{13} + 3\theta_{14} = 2\pi. \end{cases}$$

Proposition 10.2. If RC2 tiles, it must be the Sommerville No. 3.

Proof. We perform an analysis similar to that of RC1. By Lemma 9.1, if RC1 tiles, then the solid angle Ω_1 must divide 4π . From interval minimization, we obtained the intervals in which θ_{12} and θ_{13} must lie, which give the following range for Ω_1 :

$$\frac{4\pi}{\Omega_1} \in [11.9772, 12.0241]$$
.

Because the left-hand side must be an integer, it is exactly 12. Substituting this back into (3) yields

$$\theta_{14} = \frac{\pi}{3}$$
 and $\theta_{12} + \theta_{13} = \pi$.

We can now solve for θ_{12} directly using the determinant condition in Lemma 2.3, yielding

$$(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}) = \left(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\right).$$

Thus, by Table 1, RC1 is the Sommerville No. 3. The code for this computation is in the file Type_aaabcd_remaining.nb.

Remaining case 3 (RC3). RC3 belongs to the code type abaacb. It is characterized by

RC3:
$$\begin{cases} \theta_{24} = \pi/2, \\ \theta_{12} + \theta_{14} = \pi/2, \\ \theta_{13} + \theta_{34} = \pi. \end{cases}$$

Proposition 10.3. RC3 has greater normalized surface area than the Sommerville No. 1.

Proof. Mathematica can solve for dihedral angles of RC3 in closed form. By direct computation of surface area S and volume V, RC3 has normalized surface area $S/V^{2/3}$ of approximately 8.395. The normalized surface area of the Sommerville No. 1 is approximately 7.413. The code for this computation is in the file Type_abaacb_remaining.nb.

Remaining case 4 (RC4). RC4 belongs to the code type abaacd. It is characterized by

(4)
$$RC4: \begin{cases} \theta_{13} = \pi/2, \\ \theta_{24} = \pi/2, \\ \theta_{34} = \pi/3, \\ \theta_{12} + 2\theta_{23} = \pi. \end{cases}$$

Proposition 10.4. RC4 belongs to Goldberg's first infinite family.

Proof. Mathematica can solve for θ_{14} as

$$\theta_{14} = \frac{\pi - \theta_{12}}{2}.$$

By this and (4), the dihedral angles of RC4 are

$$(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}) = \left(\pi - 2\alpha, \frac{\pi}{2}, \alpha, \alpha, \frac{\pi}{2}, \frac{\pi}{3}\right).$$

By Proposition 2.7, RC4 belongs to the first Goldberg family. \Box

Remaining case 5 (RC5). RC5 belongs to the code type abaacd. It is characterized by

(5)
$$RC5: \begin{cases} \theta_{13} = \pi/2, \\ \theta_{24} = \pi/2, \\ \theta_{34} = \pi/3, \\ \theta_{12} + \theta_{14} + \theta_{23} = \pi. \end{cases}$$

Proposition 10.5. RC5 belongs to Goldberg's first infinite family.

Proof. As in the case of RC4, Mathematica can solve for θ_{14} as

$$\theta_{14} = \frac{\pi - \theta_{12}}{2}.$$

From (5), the dihedral angles of RC5 are

$$(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}) = (\pi - 2\alpha, \frac{\pi}{2}, \alpha, \alpha, \frac{\pi}{2}, \frac{\pi}{3}).$$

By Proposition 2.7, RC5 belongs to the first Goldberg family. \Box

Remaining case 6 (RC6). RC6 belongs to the code type abcard. It is characterized by

(6)
$$RC6: \begin{cases} \theta_{13} = \pi/2, \\ \theta_{34} = \pi/3, \\ \theta_{12} + \theta_{23} = \pi/2, \\ \theta_{14} + \theta_{24} = \pi. \end{cases}$$

Proposition 10.6. RC6 belongs to Goldberg's second infinite family.

Proof. The idea of the proof is to show that two copies of RC6 form a tetrahedron in Goldberg's first infinite family. Put two copies of RC6, $V_1V_2V_3V_4$ and $V_1'V_2'V_3'V_4'$, together by rotating and attaching the faces opposite vertex V_3 as in Figure 10.

Because $\theta_{14} + \theta_{24} = \pi$, V_3' lies on the same plane as V_1, V_3, V_4 and also on the same plane as V_2, V_3, V_4 . Thus V_3' lies on the same line as V_3 and V_4 . We conclude that the two copies form a tetrahedron $V_2V_3V_1V_3'$ with V_4 being the midpoint of the edge V_3V_3' . This new tetrahedron has a sextuple of dihedral angles

$$\left(\alpha,\pi-2\alpha,\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{3},\alpha\right),$$

which corresponds to Goldberg's first infinite family. By Proposition 2.7, it belongs to Goldberg's first infinite family. Because RC6 is formed by slicing a member of Goldberg's first infinite family in half along an edge with dihedral angle $\pi/3$, RC6 belongs to Goldberg's second infinite family.

Remaining case 7 (RC7). RC7 belongs to the code type abccbb. It is characterized by

(7)
$$RC7: \begin{cases} \theta_{12} = \pi/3, \\ \theta_{14} = \theta_{23} = \pi/2, \\ \theta_{13} = \theta_{24}, \\ 2\theta_{13} + \theta_{34} = \pi. \end{cases}$$

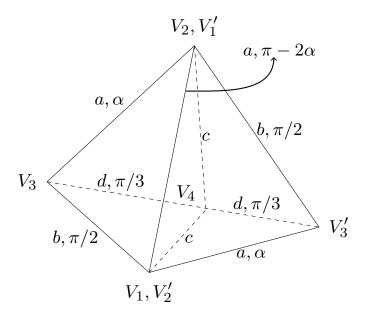


FIGURE 10. Tetrahedron constructed by rotating RC6 and attaching the faces opposite vertex V_3 , labeled with the same vertices as RC6. Note that $\theta_{14} + \theta_{24} = \pi$, so V_3' lies on the same plane V_1, V_3, V_4 and also on the same plane as V_2, V_3, V_4 .

Proposition 10.7. RC7 is a member of Goldberg's first infinite family. Proof. From (7), the dihedral angles of RC7 are

$$(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}) = (\frac{\pi}{3}, \alpha, \frac{\pi}{2}, \frac{\pi}{2}, \alpha, \pi - 2\alpha).$$

By Proposition 2.7, RC7 belongs to the first Goldberg family. \Box

We conclude Section 7 by proving that the Sommerville No. 1 has less surface area than any tile of the ten non-characterized types of Figure 9.

Theorem 10.8. The Sommerville No. 1. has less surface area than any tetrahedral tile of the ten non-characterized types of Figure 9.

Proof. By Proposition 9.5, the Sommerville No. 1 has less surface area than any tile of the ten non-characterized types of Figure 9 except possibly for the remaining cases RC1–RC7. By Proposition 10.1, RC1 does not tile. By Proposition 10.3, RC3 has greater surface area than the Sommerville No. 1. By Propositions 10.2, 10.4, 10.5, 10.6 and 10.7, the other remaining cases are among the tiles identified in previous literature; by Corollary 3.7, the Sommerville No. 1 minimizes surface area among these cases. Therefore the Sommerville No. 1 has less surface area than any tile of the ten non-characterized types of Figure 9.

11. THE BEST TETRAHEDRAL TILE

Section 11 concludes the paper and provides some conjectures about tetrahedral tiles and n-hedral tiles in general. Extension to other n-hedral tiles and non-face-to-face tiles is briefly discussed. We begin with Theorem 11.1, which states that the Sommerville No. 1 is the best tetrahedral tile.

Theorem 11.1. The least-surface-area, face-to-face tetrahedral tile of unit volume is uniquely the Sommerville No. 1.

(See Figure 1 of the Introduction.)

Proof. Proposition 6.2 classifies all tetrahedra into the 25 types of Figure 9. By Theorem 6.7, the Sommerville No. 1 uniquely minimizes surface area among the tiles of the fifteen types in the orientation-preserving, Edmonds, and $2\pi/n$ groups circled in Figure 9. By Theorem 10.8, the Sommerville No. 1 has less surface area than the tiles of the remaining ten types in Figure 9. Therefore the Sommerville No. 1 uniquely minimizes surface area among all tetrahedral tiles.

Although this paper did not completely characterize all tetrahedral tiles, we conjecture that the list of tetrahedral tiles as outlined in Section 3 is complete. Conjecture 11.2. The Sommerville and Goldberg lists of face-to-face tetrahedral tiles are complete. In particular, a tetrahedral tile has at least one dihedral angle of $\pi/2$.

We also conjecture that any area-minimizing n-hedral tile is convex and is identical to its mirror image.

Conjecture 11.3. For any n, the least-area n-hedral tile is convex.

Conjecture 11.4. For any n, the least-area n-hedral tile is congruent to its mirror image.

Remark 11.5. This is consistent with the proven tiles for n = 4, 5, 6. It is also consistent with the conjectured least-area tiles for $7 \le n \le 14$ presented by Gallagher *et al.* [GGH].

Finally, we comment on how the methods in this paper might be applicable to n-hedral tiles for n > 4 and to non-face-to-face tilings.

n-hedra. For n > 4, not all n-hedra are combinatorially equivalent. This may present more difficulty, but there are also potentially more tiling restrictions because different polygonal shapes cannot coincide in a face-to-face tile. An extra difficulty might also be that the dihedral angles no longer determine the polyhedron as in the tetrahedron case (Prop. 2.7). One of the main tools developed in this paper is the edge-length graph (Sect. 4). For general n-hedra, we can easily generalize the edge-length graph to have nodes being (n-1)-tuples. However, this loses some information, as non-triangular faces cannot be uniquely identified by edge lengths alone. Bounds on the dihedral angles in the spirit of Section 8 may also be obtained by observing that for a good isoperimetric candidate, for a given volume, the diameter cannot be large. Then for convex polyhedra, a dihedral angle cannot be very small, otherwise the volume would be close to 0. Combinatorial explosion may also be a concern: in the next unproven case, n=7, the combinatorial shape of the pentagonal prism has 15 edges. Suppose each dihedral angle is of the form $2\pi/n$ in the spirit of Theorem 5.6. If there are 20 values of n, then the number of cases could be of the

order 10^{19} . Similarly, the 2074 cases considered by code in Section 9 may increase substantially in number.

Non-face-to-face tiles. For non-face-to-face tetrahedral tiles, the linear equations have to be modified. If two edges of different lengths coincide, then the linear combinations would contain dihedral angles associated with different edge lengths, unlike those presented in Table 5. If an edge coincides with a face of an adjacent tile, then the dihedral angles will sum to π instead of 2π . Both of these effects can also occur together. The difficulty may be the classification of all tiling behaviors that arise from these effects. Imposing the assumption that the tiling is periodic may help in navigating these difficulties.

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