Proof. Poisson's equation gives us that

$$\nabla^2\phi = -4\pi G\rho$$

where $\rho(\mathbf{r})$ is the density of the matter. Applying the Divergence Theorem we have

$$\iint_{\partial R} \mathbf{f}\cdot \mathrm{d}\mathbf{S} = \iiint_R \nabla\cdot\mathbf{f}\,\mathrm{d}V = \iiint_R \nabla^2\phi\,\mathrm{d}V = -4\pi G\iiint_R \rho\,\mathrm{d}V = -4\pi GM.$$

Conversely suppose that we know

$$\iint_{\partial R} \mathbf{f}\cdot\mathrm{d}\mathbf{S} = -4\pi GM$$

for any bounded region R. Then

$$\iiint_R \nabla^2\phi\,\mathrm{d}V = -4\pi G\iiint_R \rho\,\mathrm{d}V$$

and so

$$\iiint_R \left(\nabla^2\phi + 4\pi G\rho\right)\,\mathrm{d}V = 0 \text{ for any bounded region } R.$$

Hence (at least if $\nabla^2\phi$ and ρ are piecewise continuous) we have

$$\nabla^2\phi + 4\pi G\rho \equiv 0.$$

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Solution. (to Example 132) Method Two - Gauss' Flux Theorem.
Alternatively, we may use Gauss' Flux Theorem applied to concentric spheres centred on the shell's centre.
Now, ϕ is only dependent on r and, so, is constant on the sphere $r = R$, which has surface area $4\pi R^2$.
So, if we apply the flux theorem to the region $r \leqslant R$ we have

$$\iint_{r=R} \nabla\phi\cdot\mathrm{d}\mathbf{S} = \iint_{r=R} \phi'(R)\,\mathbf{e}_r\cdot\mathrm{d}\mathbf{S} = \iint_{r=R} \phi'(R)\,\mathrm{d}S = 4\pi R^2\phi'(R) = -4\pi GM(R)$$

where $M(R)$ is the total mass within the region $r \leqslant R$. For $a \leqslant R \leqslant b$ we have

$$\begin{aligned} M(R) &= \int_{r=a}^{R}\int_{\theta=0}^{\pi}\int_{\alpha=0}^{2\pi} \frac{1}{r}r^2\sin\theta\,\mathrm{d}\alpha\,\mathrm{d}\theta\,\mathrm{d}r \\ &= 2\pi\times[-\cos\theta]_0^\pi \times \int_{r=a}^{R} r\,\mathrm{d}r \\ &= 2\pi\left(R^2 - a^2\right). \end{aligned}$$