DMAB 2642
DMPB 2642

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part B: Paper B3

Honour School of Mathematics and Philosophy Part B: Paper B3

GEOMETRY

Trinity Term 2011

Saturday, 18 June 2011, 9.30am to 12.30pm

You may submit answers to as many questions as you wish; the best two from each section will count for the total mark.

You must start a new booklet for each question you attempt. Group together the booklets for each section, and indicate on the front page of each group, the questions attempted. Attach all groups of answer booklets together with the treasury tag provided. At least one booklet must be handed in for each section.

Do not turn this page until you are told that you may do so

A. Geometry of Surfaces

1. (a) What is the *Euler characteristic* χ of a topological surface with a subdivision?

(b) A *triangulation* of a surface is a subdivision where each face has three edges and three vertices, two edges can meet only at a single vertex and two faces can meet at either a single vertex or along a common edge. Assuming the existence of a triangulation, indicate briefly how any compact connected surface can be obtained by identifying edges of a polygon in pairs.

(c) (i) If a triangulation has V vertices, E edges and F triangular faces, show that $3F = 2E$ and $E = 3(V - \chi)$.

(ii) By considering the maximum number of edges that can join two vertices, prove that

$$V \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi}).$$

(iii) What is the minimal value of V that can be achieved for projective space? Construct such a triangulation.

2. (a) State the Riemann–Hurwitz theorem for a non-constant holomorphic map from a compact connected Riemann surface X to another one Y.

(b) The Weierstrass $\wp$-function is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where the sum is over all non-zero $\omega = m\omega_1 + n\omega_2$, with m, n integers. You may assume that this series converges to a holomorphic function away from the points $z = m\omega_1 + n\omega_2$ and near zero has an expansion of the form

$$\wp(z) = z^{-2} + \sum_{k=2}^{\infty} c_k z^{2k-2}.$$

(i) Describe how this function defines a holomorphic map from a Riemann surface T with Euler characteristic zero to the Riemann surface which is the extended complex plane $\mathbb{C} \cup \{\infty\}$.

(ii) Use the Riemann–Hurwitz theorem and the fact that $\wp(-z) = \wp(z)$ to identify the zeros of $\wp'(z)$.

(c) (i) Show that the function $\wp(2z)$ also defines a holomorphic map from T to $\mathbb{C} \cup \{\infty\}$ and find its degree.

(ii) Show that $\wp(2z)$ has exactly four poles.

(iii) By considering the poles of both sides of the equation prove that

$$\wp(2z) = \frac{1}{4} \left(\frac{\wp''(z)}{\wp'(z)} \right)^2 - 2\wp(z).$$

3. (a) What is the *first fundamental form* of a surface in $\mathbb{R}^3$? Express the first fundamental form $\mathrm{d}x^2 + \mathrm{d}y^2$ of the plane in polar coordinates.

(b) State a theorem about the first fundamental form which gives a criterion for coordinate neighbourhoods on two surfaces to be isometric.

(c) Calculate the first fundamental form for

(i) the helicoid $\mathbf{r}(u, v) = au(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + v\mathbf{k}$,

(ii) a surface of revolution $\mathbf{r}(u, v) = f(u)(\cos v\,\mathbf{i} + \sin v\mathbf{j}) + u\mathbf{k}$.

Find a function $f(u)$ such that points on these surfaces have isometric coordinate neighbourhoods.

B. Algebraic Curves

4. (a) A set of $n+2$ points $p_1, \ldots, p_{n+2}$ in $\mathbb{CP}^n$ is said to be in *general position* if no $n+1$ of $p_1, \ldots, p_{n+2}$ lie in a hyperplane in $\mathbb{CP}^n$.

Prove that if $p_1, \ldots, p_{n+2}$ and $q_1, \ldots, q_{n+2}$ are in general position then there is a unique projective transformation $\tau\colon \mathbb{CP}^n \to \mathbb{CP}^n$ with $\tau(p_i) = q_i$ for $i = 1, \ldots, n+2$.

(b) Let p_1, p_2, p_3, p_4 be distinct points in $\mathbb{CP}^1$ with $p_i = [x_i, y_i]$ in projective coordinates. Define the *cross-ratio* $(p_1p_2 : p_3p_4)$ in $\mathbb{C} \setminus \{0, 1\}$ by

$$(p_1p_2 : p_3p_4) = \frac{(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2)}{(x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2)}\,.$$

Show that if $\tau\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is a projective transformation then

$$\big(\tau(p_1)\tau(p_2) : \tau(p_3)\tau(p_4)\big) = (p_1p_2 : p_3p_4)$$

for all distinct p_1, p_2, p_3, p_4 in $\mathbb{CP}^1$.

(c) Conversely, show that if $\tau\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is an injective map and

$$\big(\tau(p_1)\tau(p_2) : \tau(p_3)\tau(p_4)\big) = (p_1p_2 : p_3p_4)$$

for all distinct p_1, p_2, p_3, p_4 in $\mathbb{CP}^1$, then τ is a projective transformation.

5. Let C_λ denote the nonsingular cubic curve $y^2z = x(x-z)(x-\lambda z)$ in $\mathbb{CP}^2$ for $\lambda \in \mathbb{C}\backslash\{0,1\}$.

(a) Find a projective transformation σ of $\mathbb{CP}^2$ which takes C_λ to $C_{1/\lambda}$ and fixes $[0,1,0]$.

(b) Find a projective transformation τ which takes C_λ to $C_{1-\lambda}$ and fixes $[0,1,0]$.

(c) Find a projective transformation υ which takes C_λ to $C_{1-1/\lambda}$ and fixes $[0,1,0]$.

(d) Deduce that C_λ and C_μ are isomorphic if μ belongs to $\{\lambda, \frac{1}{\lambda}, 1-\lambda, 1-\frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}\}$.

(e) When $\lambda = -1$, so that $C_\lambda = C_{1/\lambda}$, show that $\sigma\colon C_\lambda \to C_\lambda$ is of order 4, that is, $\sigma^4 \equiv 1$, and $\sigma^k \not\equiv 1$ for $k = 1,2,3$. It is known that every nonsingular cubic curve is isomorphic as a Riemann surface to $\mathbb{C}/\Lambda$ for some lattice Λ in $\mathbb{C}$, and that every holomorphic map $\phi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is of the form $\phi : z + \Lambda \mapsto az + b + \Lambda$ for $a, b \in \mathbb{C}$ with $a\Lambda \subseteq \Lambda$. Identify the lattice Λ with $C_{-1} \cong \mathbb{C}/\Lambda$, giving brief reasons for your choice.

(f) Now let $\lambda = e^{\pi i/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, so that $\lambda = 1 - \frac{1}{\lambda} = \frac{1}{1-\lambda}$. Find the order of $\upsilon\colon C_\lambda \to C_\lambda$. Again, identify the lattice Λ in $\mathbb{C}$ with $C_\lambda \cong \mathbb{C}/\Lambda$, giving brief reasons.

6. Let C be a nonsingular algebraic curve of degree d in $\mathbb{CP}^2$. What is the genus g of C, in terms of d? Define *divisors* on C, the *degree* $\deg D$ of a divisor D, and *hyperplane divisors* H on C. State the Riemann–Roch Theorem for C.

(a) Let $f\colon C \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ be a meromorphic function with $f([x,y,z])$ not identically infinity. Show that we may choose hyperplane divisors $H_1, \ldots, H_m$ on C, where $m \geq d$ and $md > 2g-2$, such that $f \in \mathcal{L}(H_1 + \cdots + H_m)$.

(b) Use the Riemann–Roch Theorem to compute $\ell(H_1 + \cdots + H_m)$ in terms of m, d.

(c) Show that the vector space V_m of degree m homogeneous polynomials $Q(x,y,z)$ in 3 variables has dimension $\frac{1}{2}(m+1)(m+2)$.

(d) Consider meromorphic functions $f_Q : C \to \mathbb{C} \cup \{\infty\}$ of the form

$$f_Q\big([x,y,z]\big) = \frac{Q(x,y,z)}{\prod_{i=1}^m (a_i x + b_i y + c_i z)}, \qquad (1)$$

where $Q(x,y,z)$ is homogeneous of degree m and $a_i x + b_i y + c_i z = 0$ defines the hyperplane divisor H_i. Explain why $f_Q \in \mathcal{L}(H_1 + \cdots + H_m)$, and why $f_Q = f_{Q'}$ if and only if $Q = Q' + P \cdot R$, where $P(x,y,z)$ is the degree d polynomial defining C and R has degree $m-d$. Compute the dimension of the vector space of such f_Q.

(e) Deduce that every meromorphic function $f\colon C \to \mathbb{C} \cup \{\infty\}$ with $f([x,y,z])$ not identically infinity may be written in the form (1).

[*You may assume that the degree of a hyperplane divisor is* d*, and the degree of a canonical divisor is* $2g-2$.]