HOW TO BEAT YOUR WYTHOFF GAMES' OPPONENT ON THREE FRONTS

AVIEZRI S. FRAENKEL
Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot, Israel 76100

1. Wythoff Games. Let a be a positive integer. Given two piles of tokens, two players move alternately. The moves are of two types: a player may remove any positive number of tokens from a *single* pile, or he may take from both piles, say k (> 0) from one and l (> 0) from the other, provided that $|k - l| < a$. The player first unable to move is the loser, his opponent the winner. Note that passing is not allowed: each player at his turn has to remove at least one token.

We show how to beat our adversary recursively, algebraically and arithmetically. In the course of doing so we shall meet some unexpected and aesthetically pleasing relationships.

The classical Wythoff game [9] is the case $a = 1$, that is, a player taking from both piles has to take the *same* number of tokens from both. See also Coxeter [3]. This special case is reportedly played in China under the name of tsianshidsi. A pleasing presentation of tsianshidsi appears in Yaglom and Yaglom [10]. A generalization of the game in a different direction was given by Connell [2]. A generalization including both that of Connell and the one given here is included in [4]. It seems that the more interesting generalization is the one given here, and presenting it alone makes it possible to show what is going on in a more transparent manner.

We start with some notation. Game positions are denoted by (x, y) with $x \leqslant y$, where x denotes the number of tokens in one pile and y the number in the other pile. Positions from which the Previous player can win whatever move his opponent will make, are called *P-positions*, and those from which the Next player can win whatever move his opponent will make are called *N-positions*. Thus $(0,0)$ is a P-position for every a, because the first player is unable to move and so the second player wins; $(0, b)$, $b > 0$, is an N-position for every a; the Next player moves to $(0,0)$ and wins. For $a = 2$, the position $(1,3)$ is a P-position: if Next moves to $(0,3)$, $(0,2)$ or $(0,1)$, then Previous, using a move of the first type, moves to $(0,0)$ and wins. If Next moves to $(1,2)$ or to $(1,1)$, then Previous, using a move of the second type, can again move to $(0,0)$.

The set of all P-positions is denoted by P, and the set of all N-positions by N.

2. A Recursive Characterization of the *P*-Positions. A list of the first few P-positions for the case $a = 2$ is given in Table 1. The table has an interesting structure. First note that $B_n - A_n = 2n$

TABLE 1. The first few P-positions of Wythoff's game for the case $a = 2$.

n	A_n	B_n
0	0	0
1	1	3
2	2	6
3	4	10
4	5	13
5	7	17
6	8	20
7	9	23
8	11	27
9	12	30
10	14	34

Aviezri Fraenkel has been engaged in communication and computer hardware design, having received his B.Sc. and M.Sc. in electrical engineering. A Ph.D. begun in E.E. ended in 1961 in mathematics under Ernst Straus at U.C.L.A. Since 1962 he has held a position at the Weizmann Institute of Science. He also held visiting positions at a number of American Universities. His research interests lie in the theory of combinatorial games, computational complexity, combinatorial number theory and information retrieval. This and other works of his have been supported materially by his wife Shaula and their six children, one of whom studies mathematics and another considers an engineering career.

(an in general). It is probably a bit harder to notice that $A_n = \text{mex}\{A_i, B_i : i < n\}$, where, for any set S, if $\bar{S}$ denotes the complement of S with respect to the nonnegative integers, then $\text{mex}\, S = \min \bar{S} =$ least nonnegative integer not in S. (mex stands for *m*inimum *ex*cluded value. The term has been coined by John H. Conway, I believe.) Thus $\text{mex}\, \varnothing = 0$. If we define the pairs (A_n, B_n) in the indicated manner for all n, then $(A_{11}, B_{11}) = (15, 37)$, since 15 is the smallest nonnegative integer not yet in the table.

We now prove formally that the pairs (A_n, B_n) as defined above do indeed constitute the set of P-positions of the game.

THEOREM 1. $P = \cup_{i=0}^{\infty}\{(A_i, B_i)\}$, *where* $A_n = \text{mex}\{A_i, B_i : 0 \leqslant i < n\}$ *and* $B_n = A_n + an$ $(n \geqslant 0)$.

Proof. From the definition of A_n and B_n as given in the theorem it follows that if $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$, then A and B are *complementary* sets of numbers, that is, $A \cup B = Z^+$ (the set of positive integers), and $A \cap B = \varnothing$. The last equality is true since if $A_n = B_m$, then $n > m$ implies that A_n is the mex of a set containing $B_m = A_n$, a contradiction; and $n \leqslant m$ is impossible since $B_m = A_m + am \geqslant A_n + an > A_n$.

In order to prove the theorem it evidently suffices to show two things: I. A player moving from some (A_n, B_n) lands in a position not of the form (A_i, B_i). II. Given any position $(x, y) \neq (A_i, B_i)$, there is a move to some (A_n, B_n). (It is useful to note that these two conditions are also necessary: the definition of P and N implies that *all* positions reachable in one move from a P-position are N-positions; whereas at least one P-position is reachable in one move from an N-position.)

I. A move of the first type from (A_n, B_n) clearly leads to a position not of the form (A_i, B_i). Suppose that a move of the second type from (A_n, B_n) produces a position (A_i, B_i). Then $i < n$. A move of the second type satisfies $|(B_n - B_i) - (A_n - A_i)| < a$, that is, $|(n - i)a| < a$, which implies $i = n$, a contradiction.

II. Let (x, y) with $x \leqslant y$ be a position not of the form (A_i, B_i) $(i \geqslant 0)$. Since A and B are complementary, every positive integer appears exactly once in exactly one of A and B. Therefore we have either $x = B_n$ or else $x = A_n$ for some $n \geqslant 0$.

Case (i): $x = B_n$. Then move $y \to A_n$.

Case (ii): $x = A_n$. If $y > B_n$, then move $y \to B_n$. If $A_n \leqslant y < B_n$, let $d = y - x$, $m = \lfloor d/a \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$. Then move $(x, y) \to (A_m, B_m)$. This is a legal move, since:

(a) $d = y - A_n < B_n - A_n = an$, hence $m = \lfloor d/a \rfloor \leqslant d/a < n$,

(b) $y = A_n + d > A_m + d \geqslant A_m + am = B_m$,

(c) $|(y - B_m) - (x - A_m)| = |(y - x) - (B_m - A_m)| = |d - am| < a$. ■

In order to play a game such as a Wythoff game as best as possible, it suffices to compute two things: (A) the nature of the present position u (P or N); (B) a next move if u is in N. Reason: let u be an artibrary game position. If (A) shows that $u \in N$, then we know that there exists some move to a position in P. Moreover, we can use (B) to find one. If, on the other hand, (A) shows that $u \in P$, we cannot do much better than an arbitrary move while exuding an air of confidence and hoping for the best, since any position reachable in one move from a P-position is necessarily an N-position, from which our opponent can win if he knows to compute (A) and (B). Now the *statement* of Theorem 1 shows how to compute (A), since the statement constitutes a characterization of the P-positions, whereas the *proof* of Theorem 1 indicates explicitly how to compute (B).

The computation of (A) and (B) (or of (B) alone when the computation of (A) is already known) will be called a *strategy* in the sequel. Summarizing our present knowledge, we can thus say that Theorem 1 and its proof jointly constitute a recursive strategy for Wythoff games in which each P-position can be computed from the previous ones.

We close this section by briefly considering the complexity of the indicated strategy. Given a position (x, y) with $0 \leqslant x \leqslant y$, we need, for computing the next move, to construct the table recursively only up to the smallest n such that either $A_n = x$ or $B_n = x$. Since $A_n \leqslant 2n$ for every a (follows from $B_n - B_{n-1} \geqslant 2$), this computation requires only $O(x)$ comparisons of table entries with x, and $O(x)$ words of memory space. We remark that once the table has been computed and stored, it takes only $O(\log x)$ steps to locate A_n such that $x = A_n$ (or B_n such that $x = B_n$), by performing a binary search in the A_n (or B_n) sequence. Since computing (B) by the method indicated in the proof of Theorem 1 requires at most $O(\log x)$ steps, the total number of computation steps is only $O(x)$. In the next section we give a closed form for the nth P-position, which enables us to beat our adversary using an explicit rather than only an implicit recursive construction, which is at the same time computationally more efficient!

3. An Algebraic Characterization of the *P*-Positions. Let

$$\alpha = \frac{2 - a + \sqrt{a^2 + 4}}{2}, \quad \beta = \alpha + a. \tag{1}$$

α is the positive root of the quadratic equation $\xi^{-1} + (\xi + a)^{-1} = 1$. Thus α and β are irrational for every positive integer a, and satisfy $\alpha^{-1} + \beta^{-1} = 1$.

The following "folk-theorem" dates back at least to Beatty [**1**]. It has many proofs and has often been rediscovered. The proof given below seems to be one of the most elegant ones. I have heard that it is due to Ostrowski.

LEMMA 1. *Let α and β be positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$. Let $A'_n = \lfloor n\alpha \rfloor$, $B'_n = \lfloor n\beta \rfloor$ $A' = \cup_{n=1}^{\infty}\{A'_n\}$ and $B' = \cup_{n=1}^{\infty}\{B'_n\}$. Then A' and B' are complementary.*

Proof. It suffices to show that exactly one member of the sequence $\zeta = \{\alpha, \beta, 2\alpha, 2\beta, 3\alpha, 3\beta, \ldots, n\alpha, n\beta, \ldots\}$ is in the interval $[h, h+1)$ for every positive integer h. Hence it suffices to show that if $M > 1$ is any integer, then there are precisely $M - 1$ members of ζ less than M. The number of $n\alpha < M$ is $\lfloor M/\alpha \rfloor$ and the number of $n\beta < M$ is $\lfloor M/\beta \rfloor$. Thus the number of members of ζ less than M is $N = \lfloor M/\alpha \rfloor + \lfloor M/\beta \rfloor$. Now

$$\frac{M}{\alpha} - 1 < \left\lfloor \frac{M}{\alpha} \right\rfloor < \frac{M}{\alpha}, \quad \frac{M}{\beta} - 1 < \left\lfloor \frac{M}{\beta} \right\rfloor < \frac{M}{\beta}.$$

Adding, $M - 2 < N < M$. Since N is an integer, we conclude $N = M - 1$. ■

Note that $A'_0 = 0 = A_0$, $B'_0 = 0 = B_0$ and $B'_n = A'_n + an$. Moreover, $\text{mex}\{A'_i, B'_i : 0 \leqslant i < n\} = A'_n$ $(n \geqslant 0)$, since A'_n and B'_n are increasing sequences and A' and B' are complementary: if the mex were not A'_n, then A'_n would never be obtained! This shows that $A'_n = A_n$ and $B'_n = B_n$ $(n \geqslant 0)$. We have proved:

THEOREM 2. *If α and β are given by* (1), *then $P = \cup_{n=0}^{\infty}\{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)\}$.*

A strategy based on this observation can be realized as follows. Since α is irrational and $1 < \alpha < 2$,

$$x = \lfloor n\alpha \rfloor \Leftrightarrow x < n\alpha < x + 1 \Leftrightarrow \frac{x}{\alpha} < n < \frac{x+1}{\alpha} \Leftrightarrow \left\lfloor \frac{x+1}{\alpha} \right\rfloor = \left\lfloor \frac{x}{\alpha} \right\rfloor + 1,$$

where (x, y) with $x \leqslant y$ is a given game position. Therefore either $x = \lfloor n\alpha \rfloor = A_n$ where $n = \lfloor (x+1)/\alpha \rfloor$, or else, by complementarity, $x = \lfloor n\beta \rfloor = B_n$, where $n = \lfloor (x+1)/\beta \rfloor$. We have thus reduced the situation to that considered in cases (ii) and (i) in the proof of Theorem 1, and hence the strategy presented in that proof can be followed. For example, if $x = \lfloor n\alpha \rfloor = A_n$ and $y < \lfloor n\alpha \rfloor + na = \lfloor n\beta \rfloor$, then letting $m = \lfloor (y - x)/a \rfloor$, we move to $(\lfloor m\alpha \rfloor, \lfloor m\beta \rfloor) \in P$. For implementing this strategy, α has to be computed to a precision of $O(\log x)$ digits, and its storage requires $O(\log x)$ words, which is only the same order of magnitude needed for storing x itself (in binary or decimal, say).

In order to give still another unexpected way for beating our opponent, we resort to the theory of continued fractions.

4. Continued Fractions and Systems of Numeration. Let α be an irrational number satisfying $1 < \alpha < 2$. Denote its *simple continued fraction* expansion by

$$\alpha = 1 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} = [1, a_1, a_2, a_3, \ldots],$$

where the a_i are positive integers. Its *convergents* $p_n/q_n = [1, a_1, \ldots, a_n]$ satisfy the recursion

$$p_{-1} = 1,\ p_0 = 1,\ p_n = a_n p_{n-1} + p_{n-2} \qquad (n \geqslant 1)$$
$$q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2} \qquad (n \geqslant 1).$$

We do not need much more on continued fractions. The reader who wishes to read up on the theory of continued fractions may like to consult, for example, Hardy and Wright [**5**, Ch. 10], Olds [**7**] or Perron [**8**]. What we do need is the fact that every irrational number has a unique expansion into an infinite continued fraction and that conversely, every infinite continued fraction represents a unique irrational number. Moreover, we will use the fact that $\alpha = [1, \dot{a}]$ where α is given by (1) and $\dot{a}$ denotes the infinite repetition of a, and a property stated just prior to Lemma 3 below.

In the next theorem we give two systems of numeration, one based on the numerators p_i and one on the denominators q_i of the convergents of α. The two systems are called *p-system* and *q-system* in the sequel.

THEOREM 3. *Every positive integer can be written uniquely in the form*

$$N = \sum_{i=0}^{m} s_i p_i, 0 \leqslant s_i \leqslant a_{i+1}, s_{i+1} = a_{i+2} \Rightarrow s_i = 0 \quad (i \geqslant 0), \tag{2}$$

and also in the form

$$N = \sum_{i=0}^{n} t_i q_i, 0 \leqslant t_0 < a_1, 0 \leqslant t_i \leqslant a_{i+1}, t_i = a_{i+1} \Rightarrow t_{i-1} = 0 \quad (i \geqslant 1). \tag{3}$$

Note. Putting $a_i = 1$ $(i \geqslant 1)$, (2) becomes the *Fibonacci counting system*, in which all the digits s_i are 0 or 1. This is the usual binary numeration system, except that there are never two consecutive ones. This system is discussed, e.g., in Knuth [**6**, Sect. 1.2.8, Ex. 34] and in Yaglom and Yaglom [**10**].

Table 2 displays the representation of the first few positive integers in the p and q-systems for the case $a_i = 2$ $(i \geqslant 1)$.

Proof. We shall prove the result for the p-system. The proof for the q-system is very similar. Given a positive integer N, let m be the largest integer such that $p_m \leqslant N$. Write

$$\begin{aligned}
N &= s_m p_m + r_m, & 0 &\leqslant r_m < p_m\\
r_m &= s_{m-1} p_{m-1} + r_{m-1}, & 0 &\leqslant r_{m-1} < p_{m-1}\\
&\ \vdots\\
r_{i+1} &= s_i p_i + r_i, & 0 &\leqslant r_i < p_i\\
&\ \vdots\\
r_2 &= s_1 p_1 + r_1, & 0 &\leqslant r_1 < p_1\\
r_1 &= s_0 p_0.
\end{aligned}$$

TABLE 2. The representation of the first few positive integers in the p and q-systems for the case $a_i = 2$ $(i \geqslant 1)$

q_3	q_2	q_1	q_0	p_3	p_2	p_1	p_0	
12	5	2	1	17	7	3	1	n
			1				1	1
		1	0				2	2
		1	1			1	0	3
		2	0			1	1	4
	1	0	0			1	2	5
	1	0	1			2	0	6
	1	1	0		1	0	0	7
	1	1	1		1	0	1	8
	1	2	0		1	0	2	9
	2	0	0		1	1	0	10
	2	0	1		1	1	1	11
1	0	0	0		1	1	2	12
1	0	0	1		1	2	0	13
1	0	1	0		2	0	0	14
1	0	1	1		2	0	1	15
1	0	2	0		2	0	2	16
1	1	0	0	1	0	0	0	17

Thus

$$N = \sum_{i=0}^{m} s_i p_i, \tag{4}$$

that is, N is representable in the p-system. (The careful reader will note that up to this point we have not used properties of continued fractions. Thus if $1 = p_0 < p_1 < p_2 < \cdots$ is any sequence of positive integers, then the representation (4) holds. Letting, e.g., $p_i = b^i$ ($b > 1$ fixed) leads to the usual representation of N to the base b.) The digits s_i of the representation (4) satisfy

$$s_i = \frac{r_{i+1} - r_i}{p_i} < \frac{p_{i+1}}{p_i} = \frac{a_{i+1}p_i + p_{i-1}}{p_i} = a_{i+1} + \frac{p_{i-1}}{p_i} \leqslant a_{i+1} + 1,$$

and so $0 \leqslant s_i \leqslant a_{i+1}$ $(i \geqslant 0)$. (However, since $q_{-1} = 0$, we get $t_0 < a_1$ for the q-system.) Suppose that $s_i = a_{i+1}$ and $s_{i-1} \geqslant 1$. Then

$$r_i \geqslant p_{i-1} \text{ and so } r_{i+1} \geqslant a_{i+1}p_i + p_{i-1} = p_{i+1},$$

a contradiction. Hence $s_i = a_{i+1} \Rightarrow s_{i-1} = 0$ $(i \geqslant 1)$.

For proving uniqueness we need the following auxiliary result.

LEMMA 2. *Let*

$$H_{i+1} = a_{i+1}p_i + a_{i-1}p_{i-2} + \cdots + a_{k+1}p_k,$$

where $k = 0$ if i is even, $k = 1$ if i is odd. Then $H_{i+1} = p_{i+1} - 1$.

In other words, the expression H_{i+1} is the equivalent in the p-system of 99....9 in the decimal system.

Proof. We add $p_{i-1} - p_{i-1}$ to the first term, $p_{i-3} - p_{i-3}$ to the second, etc. Using $p_n = a_n p_{n-1} + p_{n-2}$ $(n \geqslant 1)$, we then get

$$H_{i+1} = (p_{i+1} - p_{i-1}) + (p_{i-1} - p_{i-3}) + \cdots \begin{cases} + (p_1 - 1) & (i \text{ even}) \\ + (p_2 - p_0) & (i \text{ odd}). \end{cases}$$

Since $p_0 = 1$, we get in both cases $H_{i+1} = P_{i+1} - 1$. ■

We now resume the proof of Theorem 3. Suppose that N has two different representations:

$$N=\sum_{i=0}^{m} s_i p_i=\sum_{i=0}^{m} u_i p_i,$$

where the digits s_i and u_i satisfy the conditions imposed in (2). Let j be the largest integer such that $s_j \neq u_j$, say $s_j > u_j$. Then

$$p_j \leqslant (s_j-u_j)p_j=\sum_{i=0}^{j-1}(u_i-s_i)p_i \leqslant \sum_{i=0}^{j-1} u_i p_i$$
$$\leqslant \sum_{i=0}^{\lfloor (j-1)/2 \rfloor} a_{j-2i}p_{j-2i-1}=p_j-1,$$

a contradiction. (The last equality in this chain is Lemma 2, and the inequality just preceding it follows from the identity

$$a_{i+1}p_i > (a_{i+1}-1)p_i + a_i p_{i-1} \quad (i \geqslant 1).) \qquad \blacksquare$$

We close this section with two definitions which will be useful in the next and final section.

(i) Relative to a simple continued fraction $\alpha=[1, a_1, a_2, \ldots]$, we define a *representation* R to be an $(m+1)$-tuple

$$R=(d_m, d_{m-1}, \ldots d_1, d_0),$$

where

$$0 \leqslant d_i \leqslant a_{i+1} \quad \text{and} \quad d_{i+1}=a_{i+2} \Rightarrow d_i=0 \quad (i \geqslant 0).$$

If it is known that $d_{i-1}=d_{i-2}=\cdots=d_0=0$, we also write $R=(d_m,\ldots,d_i)$ instead of $(d_m,\ldots,d_i,0,\ldots,0)$. The *p-interpretation* I_p of a representation $R=(d_m,\ldots,d_0)$ is the number $I_p=\sum_{i=0}^{m} d_i p_i$. The *q-interpretation* of R is the number $I_q=\sum_{i=0}^{m} d_i q_i$, provided that $d_0 < a_1$; otherwise there is no q-interpretation of R. Given any positive integer k, we say that its *p-representation* $R_p(k)$ (or *q-representation* $R_q(k)$) is $(d_m,\ldots,d_0)$ if

$$k=\sum_{i=0}^{m} d_i p_i \quad \left(\text{or} \quad k=\sum_{i=0}^{m} d_i q_i,\ d_0 < a_1\right).$$

We shall later be interested in p-interpretations of q-representations! Thus for $a_i=2$ $(i \geqslant 1)$, the decimal number 12 has q-representation 1000 (see Table 2), whose p-interpretation is 17. In other words, $I_p(R_q(12))=I_p(1000)=17$.

(ii) If $R=(d_m,\ldots,d_0)$ is any representation (which might be $R_p(k)$ or $R_q(k)$ for some positive integer k), then the representation $R'=(d_m,\ldots,d_0,0)$ is called a *left shift* of R. In other words, R' is obtained from R by shifting each digit d_i of R left by one place and inserting a zero at the right. If $R=(d_m,\ldots,d_1,d_0)$ is a representation with $d_0=0$, then the representation $R''=(d_m,\ldots,d_1)$ is called a *right shift* of R.

5. An Arithmetic Characterization of the *P*-positions. We use the new numeration system introduced in the last section to give yet another, quite different, characterization of the P-positions.

Comparing Tables 1 and 2 we notice three interesting patterns. In Theorems 4 and 5 below we show that they hold indeed, in general, in the form of the following three properties.

Property 1. The set of numbers $A_n=\operatorname{mex}\{A_i, B_i : 0 \leqslant i < n\}$ $(n \geqslant 1)$ is identical to the set of numbers with p-representations ending in an even number of zeros; and the set of numbers $B_n=A_n+an$ $(n \geqslant 1)$ is identical to the set of numbers with p-representations ending in an odd number of zeros. Thus in Table 2 $R_p(1)$ ends in an even number (zero) of zeros, and so does $R_p(7)$ (ending in two zeros). Both 1 and 7 are in the A_n-column of Table 1.

Property 2. For every $n \geqslant 1$, the p-representation of B_n is a left shift of A_n: $R_p(B_n)=R'_p(A_n)$.

(Thus $(1,3)$ of Table 1 has p-representation $(1,10)$, and $(5,13)$ has p-representation $(12,120)$.)

Property 3. Let n be any positive integer. If $R_q(n)$ ends in an even number of zeros, then $I_p(R_q(n)) = A_n$. (Thus for $a_i = 2$ $(i \geqslant 1)$, $I_p(R_q(5)) = I_p(100) = 7 = A_5$.) If $R_q(n)$ ends in an odd number of zeros, then $I_p(R_q(n)) = A_n + 1$. (Thus for the case above, $I_p(R_q(4)) = I_p(20) = 6 = A_4 + 1$.)

For proving these properties we need an auxiliary result. Let $\alpha = [1, a_1, a_2, \ldots]$ be irrational with convergents $\{p_i/q_i\}$. Let $D_i = \alpha q_i - p_i$ $(i \geqslant 1)$. From the theory of continued fractions it is known that

$$-1 = D_{-1} < D_1 < D_3 < \cdots < 0 < \cdots < D_4 < D_2 < D_0 = \alpha - 1.$$

LEMMA 3. $D_j + \sum_{i=1}^m a_{j+2i} D_{j+2i-1} = D_{j+2m}$ $(j \geqslant -1)$.

Proof. We have

$$D_j + a_{j+2} D_{j+1} = \alpha q_j - p_j + a_{j+2}(\alpha q_{j+1} - p_{j+1}) = D_{j+2}$$

Thus

$$\begin{aligned} D_j + a_{j+2} D_{j+1} &= D_{j+2}, \\ D_{j+2} + a_{j+4} D_{j+3} &= D_{j+4}, \\ &\vdots \\ D_{j+2m-2} + a_{j+2m} D_{j+2m-1} &= D_{j+2m}. \end{aligned}$$

Adding proves the assertion. ■

The proof of Property 3 follows from the next theorem.

THEOREM 4. *Let $\alpha = [1, a_1, a_2, \ldots]$ be irrational with convergents $\{p_i/q_i\}$. Let n be a positive integer. If $R_q(n) = (d_m, \ldots, d_{2k})$ $(d_{2k} \neq 0, k \geqslant 0)$, then $I_p(R_q(n)) = \lfloor n\alpha \rfloor$. (That is,*

$$n = \sum_{i=2k}^{m} d_i q_i \Rightarrow \lfloor n\alpha \rfloor = \sum_{i=2k}^{m} d_i p_i.)$$

If $R_q(n) = (d_m, \ldots, d_{2k+1})$ $(d_{2k+1} \neq 0, k \geqslant 0)$, then $I_p(R_q(n)) = \lfloor n\alpha \rfloor + 1$. (That is,

$$\begin{aligned} n = \sum_{i=2k+1}^{m} d_i q_i \Rightarrow \lfloor n\alpha \rfloor &= -1 + \sum_{i=2k+1}^{m} d_i p_i \\ &= \sum_{i=0}^{k} a_{2i+1} p_{2i} + (d_{2k+1} - 1) p_{2k+1} + \sum_{i=2k+2}^{m} d_i p_i, \end{aligned}$$

where the last equality follows from Lemma 2.)

Proof. For the first case it suffices to show that

$$0 < n\alpha - \sum_{i=2k}^{m} d_i p_i < 1,$$

that is,

$$0 < \sum_{i=2k}^{m} d_i D_i < 1.$$

By Lemma 3,

$$\sum_{i=2k}^{m} d_i D_i \geqslant D_{2k} + \sum_{i=1}^{m} a_{2k+2i} D_{2k+2i-1} = D_{2k+2m} > 0,$$

$$\sum_{i=2k}^{m} d_i D_i \leqslant \sum_{i=1}^{m} a_{2k+2i-1} D_{2k+2i-2} = D_{2k+2m-1} - D_{2k-1}$$
$$\leqslant D_{2k+2m-1} + 1 < 1.$$

For the second case it suffices to show that

$$-1 < \sum_{i=2k+1}^{m} d_i D_i < 0.$$

Again by Lemma 3,

$$\sum_{i=2k+1}^{m} d_i D_i \geqslant \sum_{i=1}^{m} a_{2k+2i} D_{2k+2i-1} = D_{2k+2m} - D_{2k}$$
$$\geqslant -D_{2k} \geqslant -D_0 = 1 - \alpha > -1,$$
$$\sum_{i=2k+1}^{m} d_i D_i \leqslant D_{2k+1} + \sum_{i=1}^{m} a_{2k+2i+1} D_{2k+2i} = D_{2k+2m+1} < 0. \quad \blacksquare$$

We now prove Property 2.

THEOREM 5. *Let $\alpha = [1, \dot{a}]$, $\beta = \alpha + a$, where a is any positive integer. Then for every positive integer n, $R_p(\lfloor n\beta \rfloor) = R_p'(\lfloor n\alpha \rfloor)$.*

Proof. We have $\lfloor \alpha \rfloor = 1 = p_0$, $\lfloor \beta \rfloor = 1 + a = p_1$; hence the claim holds for $n = 1$. Suppose it holds for all $k < n$. Now $R_p(\lfloor n\alpha \rfloor)$ ends in an even number of zeros by Theorem 4. Let R' be the left shift of $R_p(\lfloor n\alpha \rfloor)$. By the induction hypothesis, $I_p(R') \neq \lfloor k\beta \rfloor$, $k < n$. In fact, $I_p(R')$ is the smallest number with representation R' ending in an odd number of zeros not yet obtained. If $I_p(R') \neq \lfloor n\beta \rfloor$, then $I_p(R')$ can never be obtained for $k > n$, since the sequence $\lfloor k\beta \rfloor$ is increasing, in contradiction to Lemma 1. ■

Theorem 4 asserts that $R_p(\lfloor n\alpha \rfloor)$ ends in an even number of zeros for all n. Theorem 5 implies, in particular, that $R_p(\lfloor n\beta \rfloor)$ ends in an odd number of zeros for all n. Since the sequences $\lfloor n\alpha \rfloor$ and $\lfloor n\beta \rfloor$ are complementary, every positive integer k such that $R_p(k)$ ends in an even (odd) number of zeros has the form $\lfloor n\alpha \rfloor$ ($\lfloor n\beta \rfloor$). This proves Property 1.

Now suppose we are given a position (x, y) with $0 < x \leqslant y$. To obtain a strategy based on Theorems 4 and 5, we compute $R_p(x)$. If it ends in an odd number of zeros, then $x = B_k$ for some k, and a winning move is $(x, y) \to (I_p(R_p''(x)), x) \in P$. If $R_p(x)$ ends in an even number of zeros, then $x = A_k$ for some k. If $y > I_p(R_p'(x))$, then the move $(x, y) \to (x, I_p(R_p'(x))) \in P$ is a winning move. If $y = I_p(R_p'(x))$, then $(x, y) \in P$, so we cannot win when starting from the given position (x, y). Finally, if $x \leqslant y < I_p(R_p'(x))$, then let $m = \lfloor (y - x)/a \rfloor$. If $R_q(m)$ ends in an even number of zeros, then $I_p(R_q(m)) = A_m$ by Theorem 4. If $R_q(m)$ ends in an odd number of zeros, then $I_p(R_q(m)) = A_m + 1$. In either case, a winning move is $(x, y) \to (A_m, A_m + ma) \in P$.

In order to estimate the complexity of this algorithm note that for $\alpha = \alpha_1 = [1, \dot{a}]$, the solution of the recursion $p_{-1} = 1$, $p_0 = 1$, $p_n = ap_{n-1} + p_{n-2}$ $(n \geqslant 1)$ is

$$p_n = \frac{1}{\alpha_1 - \alpha_2}\left(\alpha_1(\alpha_1 + a - 1)^{n+1} - \alpha_2(\alpha_2 + a - 1)^{n+1}\right)$$
$$= \frac{1}{\sqrt{a^2+4}}\left(\alpha_1\left(\frac{a + \sqrt{a^2+4}}{2}\right)^{n+1} - \alpha_2\left(\frac{a - \sqrt{a^2+4}}{2}\right)^{n+1}\right),$$

where

$$\alpha_1 = \frac{2 - a + \sqrt{a^2+4}}{2}, \qquad \alpha_2 = \frac{2 - a - \sqrt{a^2+4}}{2}$$

are the two roots of the quadratic equation $\xi^{-1} + (\xi + a)^{-1} = 1$. Since $-1 < \alpha_2 + a - 1 < 0$, we have $p_n \sim cg^{n+1}$, where $c = \alpha_1/\sqrt{a^2+4}$ and $g = (a + \sqrt{a^2+4})/2$.

Let n be the largest integer such that $x \geqslant p_n$. Since $p_n \sim cg^{n+1}$ we see that $n = O(\log x)$ steps suffice to compute $R_p(x)$ and $O(\log x)$ words of memory suffice to store it, since for computing $R_p(x)$ we need only the first $O(\log x)$ values of p_i. For the case $A_n = x \leqslant y < B_n$, since $m = \lfloor (y-x)/a \rfloor < n$, the computation of $R_q(m)$ also requires at most $O(\log x)$ steps and that much memory space. Since $n \sim \log_g(x/c)$ and g increases with a, it is seen that for large a this strategy implementation is more efficient than even the algebraic one of the previous section.

Acknowledgement. The two referees unanimously and independently purged a second part of this paper which constituted the author's sole motivation for writing the article in the first place! Nevertheless (and since both referees unanimously and independently recommended publishing the second part elsewhere), the author wishes to thank them and the editor for their very useful, detailed and constructive comments which helped to improve the article in various ways.

References

1. S. Beatty, Problem 3173, this MONTHLY, 33 (1926) 159; 34 (1927) 159.
2. I. G. Connell, A generalization of Wythoff's game, Canad. Math. Bull., 2 (1959) 181–190.
3. H. S. M. Coxeter, The golden section, Phyllotaxis and Wythoff's game, Scripta Math., 19 (1953) 135–143.
4. A. S. Fraenkel and I. Borosh, A generalization of Wythoff's game, J. Comb. Theory, 15 (1973) 175–191.
5. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 4th ed., 1960.
6. D. E. Knuth, The Art of Computer Programming, vol. 1: Fundamental Algorithms, Addison-Wesley, Reading, MA, second printing, 1969.
7. C. D. Olds, Continued Fractions, Random House, New York, 1963.
8. O. Perron, Die Lehre von den Kettenbrüchen, Band I, Teubner, Stuttgart, 1954.
9. W. Wythoff, A modification of the game of Nim, Nieuw Arch. Wisk., 7 (1907) 199–202.
10. A. M. Yaglom and I. M. Yaglom, Challenging Mathematical Problems with Elementary Solutions, translated by J. McCawley, Jr., revised and edited by B. Gordon, vol. 2, Holden-Day, San Francisco, 1967.

MISCELLANEA

SONNET

76.

R. P. BOAS

Lines written after reading too many abstracts of talks at a Mathematics meeting (after Shakespeare, Sonnet 130, "My mistress' eyes are nothing like the sun")

No diagrams within my work commute;
Language will do. Against the tide of groups,
Lie, semisimple—I'm with King Canute.
Let others prate of posets and of loops,
Functors and morphisms, maximal ideals;
Give me the clichés of an earlier age.
Let no nonstandard models of the reals,
Sur- or bijections decorate my page.
The complex plane contains enough; for me
No sheaves of germs upon a manifold.
I'll never be approved by Bourbaki;
Words grow apace, but still my soul's not sold.
And yet I think my work was as profound
As this, tricked out with terms of modish sound.