SPECTRAL THEORY

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Contents

Familiarity with basic topics in functional analysis is expected: Banach spaces, Hilbert spaces, bounded linear operators, algebraic inverse vs topological inverse, dual spaces, Hahn-Banach theorem and its consequences. Chapter 1, 2, 4.2, 6, 7.2 of lecture note of Oxford course B4.1 Functional Analysis I will be enough. Please see `https://courses.maths.ox.ac.uk/pluginfile.php/105425/mod_resource/content/22/B4.1LectureNotes.pdf`.

We will basically follow the material from the last bit of Cambridge Part III course Functional Analysis, for details see `https://minterscompactness.wordpress.com/wp-content/uploads/2018/09/functional-analysis-part-iii-notes.pdf`.

1. Introduction

Let H be a Hilbert space and T be a compact, self-adjoint operator on H. Then we have a spectrum decomposition of T. As T is compact, $\sigma(T) \setminus \{0\} = \sigma_P(T)$ is countable and $0 \in \overline{\sigma_P(T)}$. Let $\sigma_P(T) = \{\lambda_1, \lambda_2, \cdots\}$ and E_k be the associated eigenspace of λ_k. Then

$$Tx = \sum_{k=1}^{\infty} \lambda_k P_{E_k} x$$

where P_{E_k} is the orthogonal projection on E_k, and the convergence is in the sense of operator norm. Being slightly crazy, if P_{E_k} can be thought as some "measure" on some "sets", i.e., $P_{E_k} = m(F_k)$ for some F_k inside a single, say, compact Hausdorff topological space, then we can write

$$T = \sum_{k=1}^{\infty} \lambda_k P_{E_k} = \sum_{k=1}^{\infty} \lambda_k m(F_k) = \int \sum_{k=1}^{\infty} \lambda_k \mathbf{1}_{F_k} \, \mathrm{d}m.$$

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This is where *Boerl Functional Calculus* comes from. Indeed we can use this intuition to define a more general version of spectral theories, which we will give an introduction in this note.

2. Banach and C^*-algebra

Let $\mathcal{B}(H)$ denotes the space of bounded linear operators on H. Then by Riesz representation theorem, for each $T \in \mathcal{B}(H)$ there exists a unique $T^* \in \mathcal{B}(H)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$. Now we have some simple properties of T and T^*:

(1) $(\lambda T + \mu S)^* = \bar{\lambda}T^* + \bar{\mu}S^*$;
(2) $(TS)^* = S^*T^*$;
(3) $T^{**} = T$;
(4) $\|T^*T\| = \|T\|^2$.

We also survey some general properties for $\mathcal{B}(H)$: For $T, S \in \mathcal{B}(H)$, we have $TS \in \mathcal{B}(H)$ and $\|TS\| \leq \|T\|\|S\|$. Now we abstract everything out, get rid of H and arrive at the definitions:

Definition 2.1. *An* algebra A *over a field* $\mathbb{F}$ *is a vector space equipped with addition* $+$*, vector multiplication* $\times$ *and scalar multiplication* $\cdot$ *such that*

(1) $\forall a, b, c \in A$, *we have* $a \times (b \times c) = (a \times b) \times c$*;*
(2) $\forall a, b \in A, \lambda \in \mathbb{F}$, *we have* $\lambda \cdot (a \times b) = (\lambda \cdot a) \times b = a \times (\lambda \cdot b)$*;*
(3) $\forall a, b, c \in A$, *we have* $a \times (b + c) = a \times b + a \times c$.

We also notice that there is an identity in $\mathcal{B}(H)$, so we should separate those algebras with identity out.

Definition 2.2. *An* unital *algebra is an algebra* A *with an element* $1 \neq 0$ *in* A *such that* $\forall a \in A$, *we have* $1 \cdot a = a \cdot 1 = a$.

We always want a norm to study infinite dimensional spaces.

Definition 2.3. *An* algebra norm *on an algebra* A *is a vector space norm* $\|\cdot\| : A \to \mathbb{R}$ *such that* $\|ab\| \leq \|a\|\|b\|$ $\forall a, b \in A$. *The pair* $(A, \|\cdot\|)$ *is called a normed algebra. In particular this ensures vector multiplication is continuous.*

An unital *normed algebra is a normed algebra* A *which is also a unital algebra and satisfies* $\|1\| = 1$.

A Banach Algebra *is a complete, normed algebra. We will abbreviate it as BA.*

We finally arrive at the definition of a C^*-algebra.

Definition 2.4. *An* involution *is a map* $^* : A \to A$, *sending* $x \mapsto x^*$, *such that:*

(1) $(\lambda x + \mu y)^* = \bar{\lambda}x^* + \bar{\mu}y^*$*;*
(2) $(xy)^* = y^*x^*$*;*
(3) $x^{**} = x$.

for all $x, y \in A, \lambda, \mu \in \mathbb{C}$.

Definition 2.5. *A* C^*-algebra *is a Banach algebra with an involution* * *such that the* C^*-equation *holds:*

$$\|x^*x\| = \|x\|^2 \quad \forall x \in A.$$

We have the abstract versions of different operators:

Definition 2.6. *We say:*

(1) x *is* self-adjoint *(Hermitian) if* $x^* = x$*;*

(2) x *is* unitary *if A is unital, and $x^*x = xx^* = 1$;*
(3) x *is* normal *if $x^*x = xx^*$.*

Definition 2.7. *A* homomorphism θ *between two algebras A and B is a linear map such that $\theta(xy) = \theta(x)\theta(y)$. If A, B are unital with units 1_A and 1_B respectively, then θ is a* unital homomorphism *if $\theta(1_A) = 1_B$ as well.*

Definition 2.8. *A $*$-homomorphism between C^*-algebras A, B is a homomorphism $\theta : A \to B$ such that $\theta(x^*) = (\theta(x))^*$. A $*$-isomorphism is a bijective $*$-homomorphism.*

3. Borel Functional Calculus

Let K be a compact, Hausdorff topological space and $\mathcal{B}_K$ be the Borel sigma algebra on K. we define our *vector-valued measure* on $\mathcal{B}_K$.

Definition 3.1. *A* resolution of the identity of H over K *is a map $P : \mathcal{B}_K \to \mathcal{B}(H)$ such that:*

(1) *$P(\emptyset) = 0$ and $P(K) = I_H$;*
(2) *$P(E)$ is an orthogonal projection $E \in \mathcal{B}_K$;*
(3) *$P(E \cap F) = P(E)P(F)$ for any $E, F \in \mathcal{B}_K$;*
(4) *If $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F)$ (not necessarily countably additive);*
(5) *$\forall x, y \in H$, the map $P_{x,y} : \mathcal{B}_K \to \mathbb{C}$ given by:*

$$P_{x,y}(E) := \langle (P(E))(x), y \rangle$$

is a regular Borel measure on K.

When doing integration, we consider an analogous of L^∞ spaces:

Definition 3.2. *A Borel function $f : K \to \mathbb{C}$ is P-*essentially bounded *if there exists a Borel set E with $P(E) = 0$ such that f is bounded on $K \setminus E$. We define*

$$L^\infty(P) := \{f : f : K \to \mathbb{C} \text{ is Borel and } P\text{-essentially bounded}\}$$

equipped with norm

$$\|f\|_\infty := \inf\left\{ \|f\|_{K\setminus E} := \sup_{K\setminus E} |f| : E \in \mathcal{B}_K \text{ such that } P(E) = 0 \right\}.$$

Let $L^\infty_s(P)$ be the subspace of simple functions in $L^\infty(P)$.

We firstly consider the integral on simple functions: Let s be a simple function. WLOG we can suppose $(E_i)_{1\le i\le m}$ is a Borel partition of K and $s = \sum_{i=1}^m a_i \mathbf{1}_{E_i}$ (why?). Define

$$\int_K s \, \mathrm{d}P = \sum_{i=1}^m a_i P(E_i).$$

We have to check that it is well-defined. Let $s = \sum_{i=1}^n b_i \mathbf{1}_{F_i}$ be another representation of s, where $(F_i)_{1\le i\le n}$ is another Borel partition of K. As they both agree we have, by considering $E_i \cap F_j$, that either $a_i = b_j$ or $P(E_i \cap F_j) = 0$. Hence

$$a_i P(E_i \cap F_j) = b_j P(E_i \cap F_j).$$

By using the identity

$$P(E_i) = P(E_i \cap K) = P\left(E_i \cap \bigcup_{j=1}^{n} F_j\right) = \sum_{j=1}^{n} P(E_i \cap F_j),$$

we have

$$\sum_{i=1}^{m} a_i P(E_i) = \sum_{i,j} a_i P(E_i \cap F_j) = \sum_{i,j} b_j P(E_i \cap F_j) = \sum_{j=1}^{n} b_j P(F_j).$$

We have some useful facts about such integral

Proposition 3.3. *The map*

$$\Phi : L_s^\infty(P) \to \mathcal{B}(H), \quad L_s^\infty(P) \ni s \to \int_K s \ \mathrm{d}P$$

is an isometric $*$-homomorphism such that

(1) $\Phi(\mathbf{1}_k) = I_H$;
(2) $\langle \Phi(s)(x), y\rangle = \int_K s \ \mathrm{d}P_{x,y}$;
(3) $\|\Phi(s)(x)\|^2 = \int_K |s|^2 \ \mathrm{d}P_{x,x}$.

Proof. (1) is trivial. We check it is an isometric injective $*$-homeomorphism first. Linearity could be check with some careful arguments like above. Now as $P(E_i)$ are orthogonal projections we have $P(E_i)^* = P(E_i)$ (why?), so

$$(\Phi(s))^* = \left(\sum_{i=1}^{m} a_i P(E_i)\right)^* = \sum_{i=1}^{n} \overline{a_i} P(E_i) = \Phi(\bar{s}).$$

We also notice that if $s = \sum_{i=1}^m a_i \mathbf{1}_{E_i}$ and $t = \sum_{i=1}^n b_i \mathbf{1}_{F_i}$ we have $st = \sum_{i,j} a_i b_j \mathbf{1}_{E_i \cap F_j}$ and hence

$$\Phi(st) = \sum_{i,j} a_i b_j P(E_i \cap F_j) = \Phi(st) = \sum_{i,j} a_i b_j P(E_i) P(F_j) = \Phi(s)\Phi(t).$$

Thus indeed Φ is a $*$-homeomorphism. Now simply calculate

$$\langle \Phi(s)x, y\rangle = \sum_{i=1}^{m} a_i \langle P(E_i)x, y\rangle = \sum_{i=1}^{m} a_i P_{x,y}(E_i) = \int_K s \ \mathrm{d}P_{x,y},$$

so (2) holds. For (3) set $y = \Phi(s)(x)$ we have

$$P_{x,\Phi(s)(x)}(E_i) = \left\langle P(E_i)x, \sum_{i=1}^{m} a_i P(E_i)(x)\right\rangle = \overline{a_i}\langle P(E_i)x, P(E_i)x\rangle = \overline{a_i} P_{x,x}(E_i)$$

where we have used the fact that $P(E_i)$ are self-adjoint and $P(E_i \cap E_j) = 0$. Now we check the isometry. By (3) we see that

$$\|\Phi(s)(x)\|^2 = \int_K |s|^2 \ \mathrm{d}P_{x,x} = \sum_{i=1}^{n} |a_i|^2 \langle P(E_i)x, x\rangle \leq \sup_i |a_i|^2 \|x\|^2.$$

Hence, $\|\Phi(s)\| \leq \|s\|_\infty = \sup_i |a_i|$. As $i = 1, 2, \cdots, m$ is finite, $\sup_i |a_i|$ is attended by sum a_j where E_j is non-null. Take $x \in \mathrm{Im}(P(E_j))$ we have $\|\Phi(s)(x)\| = |a_j|\|x\|$ so $\|\Phi(s)\| = \|s\|_\infty$. □

As $L_s^\infty(P)$ is dense in $L^\infty(p)$, by isomstry property we can extend Φ to $L^\infty(P)$:

Proposition 3.4. *(Borel Functional Calculus) There exists a unique map*

$$\Phi : L^\infty(P) \to \mathcal{B}(H), \quad L^\infty(P) \ni f \to \int_K f \, \mathrm{d}P$$

is an isometric $*$-homomorphism such that

(1) $\Phi(\mathbf{1}_k) = I_H$;
(2) $\langle \Phi(f)(x), y \rangle = \int_K f \, \mathrm{d}P_{x,y}$;
(3) $\|\Phi(f)(x)\|^2 = \int_K |f|^2 \, \mathrm{d}P_{x,x}$;
(4) *$S \in B(H)$ commutes with every $\Phi(f)$ if and only if S commutes with $P(E)$ for every $E \in \mathcal{B}_K$.*

Proof. We only need to check uniqueness and (4). For (4), If S commutes with every $P(E)$ then by taking approximation by simple functions, we have S commutes with every $\Phi(f)$. The other direction is trivial. For uniqueness, let Ψ be another such map, then

$$\langle \Phi(f)(x), y \rangle = \int_K f \, \mathrm{d}P_{x,y} = \langle \Psi(f)(x), y \rangle$$

for any $x, y \in H$, By taking $y = \Phi(f)(x) - \Psi(f)(x)$ we are done. □

4. Spectral Theory

4.1. **Spectrums.** Now we would like to talk about invertibility of some elements in a Banach algebra or C^*-algebra A. In particular we can define the spectrum of any element $x \in A$ be exactly the same as how we define it for bounded linear operators.

Definition 4.1. *For an unital algebra A and $x \in A$ we define the* spectrum $\sigma(x)$ *as*

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda \text{ is not invertible}\}.$$

In here we are going to develop a very general notion of spectrum and prove the corresponding spectral theories under these notions. In the following, unless specifically noted, A denotes a unital Banach algebra.

Lemma 4.2. *If $\|1 - a\| < 1$, then a is invertible and $\|a^{-1}\| \leq \frac{1}{1 - \|1-a\|}$.*

Proof. Let $x = 1 - a$, then $\sum_{i=1}^{\infty} x^i$ is the explicit inverse of a, which converges as it converges absolutely. □

From this we can easily show that if $G(A)$ denotes the invertible elements of A, then $G(A)$ is open.

Theorem 4.3. *Let $x \in A$, then $\sigma(x)$ is a non-empty, compact and is contained in $B_{\|x\|}(0) = \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$.*

Proof. By Lemma 4.2 we have $x - \lambda$ is invertible if $\lambda > \|x\|$, so the last part is proven. If $x - \lambda$ is invertible, then for any μ such that $|\mu - \lambda| < \|x - \lambda\|$, $\mu - x$ is invertible. So $\mathbb{C} \setminus \sigma(x)$ is open and hence $\sigma(x)$ is closed and bounded, hence compact.

The only non-trivial part is the non-emptiness. Suppose that $\sigma(x)$ is empty. Then for any bounded linear functional f on A, $g_f : \lambda \to f\left((x - \lambda)^{-1}\right)$ is a holomorphic function on $\mathbb{C}$ (why?). Since for $\lambda > 2\|x\|$,

$$\|(x - \lambda)^{-1}\| \leq \frac{1}{|\lambda|} \sum_{i=0}^{\infty} \left\| \frac{x}{\lambda} \right\|^i \leq \frac{1}{\|x\|},$$

we have g_f is bounded, and hence constant by Louville's theorem. This implies that $(x-\lambda)^{-1}$ is constant over all λ, which is impossible. □

We also have the spectrum radius

Definition 4.4. *The* spectrum radius *of $x \in A$, $r(x)$, is defined as*

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

We note a formula for spectrum radius that we do not prove here:

Theorem 4.5. *(Gelfand's formula)*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_n \|x^n\|^{1/n}$$

for any $x \in A$.

4.2. **Gelfand Transform.** We notice that it is possible to define the "dual" space on a Banach algebra:

Definition 4.6. *A* character *on an algebra A is a non-zero homomorphism $A \to \mathbb{C}$. Let Φ_A denotes the set of all characters on A.*

Proposition 4.7. *Let $\varphi \in \Phi_A$. Then φ is continuous and $\|\varphi\| = 1$.*

Proof. Given $x \in A$, suppose $|\varphi(x)| > \|x\|$. Then we have $\|x/\varphi(x)\| < 1$, and so by Lemma 4.2, we have $1 - x/\varphi(x) \in G(A)$. Let $z \in A$ be the inverse of $1 - x/\varphi(x)$. Then apply φ to this expression, we have

$$1 = \varphi(1) = \varphi(z) \cdot \underbrace{\varphi(1 - x/\varphi(x))}_{=0},$$

contradiction. Hence $|\varphi(x)| \leq \|x\|$. Take $x = 1$ we have $\|\varphi\| = 1$. □

Lemma 4.8. *Let I be a proper ideal of A. Then $\overline{I}$ is a proper ideal of A.*

Proof. As I is proper, $I \cap G(A) = \emptyset$. As $G(A)$ is open, $\overline{I} \cap G(A) = \emptyset$. By continuity of multiplication and addition, $\overline{I}$ is an ideal hence proper. □

Let $\mathcal{M}_A$ be the collection of all maximal ideals of A.

Proposition 4.9. *Let A be commutative, then its maximal ideals are exactly the kernel of its characters.*

Proof. For each φ, $\ker(\varphi)$ is a maximal ideal. Since $\varphi \not\equiv 0$, $\ker(\varphi)$ has co-dimension 1. So it follows that $\ker(\varphi)$ is maximal.

Assume that $\ker(\varphi) = \ker(\psi)$, for some $\varphi, \psi \in \Phi_A$. Then given $x \in A$, we have: $x - \varphi(x) \in \ker(\varphi) = \ker(\psi)$. Hence $\psi(x - \varphi(x)) = \psi(x) - \varphi(x) = 0$. Hence $\psi = \varphi$.

Let M be a maximal ideal, then A/M is a field and it is isomorphic to $\mathbb{C}$ (topologically, check why) and the quotient map $A \to A/M$ defines a character. □

Theorem 4.10. *Let A be commutative and let $x \in A$. Then:*

1. $x \in G(A) \Leftrightarrow x \notin \ker(\varphi)$ *for any character* φ;
2. $\sigma(x) = \{\varphi(x) : \varphi \in \Phi_A\}$;
3. $r(x) = \sup\{|\varphi(x)| : \varphi \in \Phi_A\}$.

Proof. (1) follows from proposition 4.9.

For (2), let $\lambda \in \sigma(x)$ and then by (1) there exists character φ such that $\lambda - x \in \ker(\varphi)$, so $\lambda = \varphi(x)$. If $\lambda = \varphi(x)$ for some $\varphi \in \Phi_A$ then $\lambda - x \in \ker(\varphi)$ so it is not invertible by (1).

(3) follows immediately from (2). □

Exercise 4.11. *Prove the following formulas: Suppose A is not necessarily commutative but x, y commutes, then we have*

$$r(x+y) \leq r(x) + r(y) \quad \text{and} \quad r(xy) \leq r(x)r(y).$$

Equip Φ_A with the topology that, $\Phi_A \ni f_n \to f$ if and only if $f_n(x) \to f(x)$ for any $x \in A$. This is called the *weak* * topology (w^*) of Φ_A, which makes (Φ_A, w^*) into a compact, Hausdorff space (Banach-Alaoglu Theorem). Then we see as spectrum can be represented by characters, we have the natural identification:

Definition 4.12. (Φ_A, w^*) *is called the* spectrum *of A. The map $\hat{x} : \Phi_A \to \mathbb{C}$ defined by $\hat{x}(\varphi) = \varphi(x)$ is called the* Gelfand transform *of x. The map $x \to \hat{x}$ is called the* Gelfand map.

By Theorem 4.10 we see that the image of $\hat{x}$ is $\sigma(x)$, and $\hat{x}$ is a continuous map on Φ_A. We note that $C(\Phi_A)$ can be identified as a Banach algebra equipped with the supremum norm $\|f\|_\infty = \sup_{x \in \Phi_A} |f(x)|$.

Theorem 4.13. *(Gelfand Representation Theorem) The Gelfand map is a continuous unital homomorphism, and:*

1. $\|\hat{x}\|_\infty = r(x)$;
2. $\sigma(\hat{x}) = \sigma(x)$;
3. $\hat{x} \in G(C(\Phi_A)) \Leftrightarrow x \in G(A)$.

Proof. To check homomorphism is easy, and (1) follows immediately from Theorem 4.10 (3), and boundness follows.

For (2), we note that the spectrum of a continuous function on a compact set is just the image of it (why?), so $\sigma(\hat{x}) = \operatorname{Im}(\hat{x}) = \sigma(x)$.

For (3), $\hat{x} \in G(C(\Phi_A))$ if and only if $0 \notin \sigma(\hat{x})$, if and only if $0 \notin \sigma(x)$, if and only if x is invertible. □

For the case of a C^*-algebra, their characters behave better:

Theorem 4.14. *(Commutative Gelfand-Naimark Theorem). Let A be a commutative, unital C^*-algebra. Then, the Gelfand map $x \mapsto \hat{x}$, is an isometric $*$-isomorphism between A and $C(\Phi_A)$. This means all C^*-algebras are $C(K)$ for some compact, Hausdorff space K.*

Proof. We proof an intuitive lemma first

Lemma 4.15. *Let A be an unital C^*-algebra. Then characters on A are $*$-homomorphisms.*

Proof. We check $\varphi(x^*) = \overline{\varphi(x)}$. Suppose x is self-adjoint, and $\varphi(x) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. Then for $t \in \mathbb{R}$, let $z_t = x + it$. Then:

$$|\varphi(z_t)|^2 = |\alpha + i(\beta + t)|^2 = \alpha^2 + (\beta + t)^2.$$

But we also have

$$|\varphi(z_t)|^2 \leq \|z_t\|^2 = \|z_t^* z_t\| = \|(x - it)(x + it)\| = \|x^2 + t^2\| \leq \|x\|^2 + t^2$$

So hence we see

$$\alpha + 2\beta t + \beta^2 \leq \|x\|^2$$

and this is true for all t. Hence we must have $\beta = 0$ and that $\varphi(x) \in \mathbb{R}$.

Now assume x is arbitrary. Then we can write $x = h + ik$, where h, k are self-adjoint (why?). Then $x^* = h - ik$ and

$$\varphi(x^*) = \varphi(h) - i\varphi(k) = \overline{(\varphi(h) + i\varphi(k))} = \overline{\varphi(x)}$$

where we used $\varphi(h), \varphi(k) \in \mathbb{R}$ by the self-adjoint case above. □

We check that the Gelfand map is a isometric surjective $*$-homomorphism. Notice that $\hat{x^*}(\varphi) = \varphi(x^*) = \overline{\varphi(x)} = \hat{x}(\varphi)^*$, so indeed it is a $*$-homomorphism.

For isometric we see immediately as $\|\hat{x}\|_\infty = r(x) = \|x\|$. The last equality follows, as A is commutative and hence x is always normal, and for normal element we have $\|x\| = r(x)$ by using Gelfand formula.

For surjective, we see that the image of Gelfand map is a closed unital $*$-subalgebra that seperate points. Hence by Stone-Weiestrass theorem (A unital $*$-subalgebra of $C(K)$ that separates point is dense in $C(K)$, for any compact Hausdorff space K) we see that it is $C(\Phi_A)$. □

Corollary 4.16. *Let A be a unital C^*-algebra. Let $x \in A$ and $\varphi \in \Phi_A$.*

$$x \text{ self-adjoint } \Rightarrow \varphi(x) \in \mathbb{R}$$
$$x \text{ unitary } \Rightarrow \varphi(x) \in S^1.$$

We can now define the spectrum properties of self-adjoint and unitary elements. In the following, if $x \in A$ and $x \in B \subset A$ be a subalgebra of A, then we define $\sigma_B(x)$ to be the spectrum of x only consider B.

Proposition 4.17. *Let A be a unital C^*-algebra, and let $x \in A$. Then we have:*

(1) *x Self-adjoint $\Rightarrow \sigma_A(x) \subset \mathbb{R}$;*
(2) *x unitary $\Rightarrow \sigma_A(x) \subset S^1$.*
(3) *If B is a unital C^*-subalgebra of A and $x \in B$ is normal, then $\sigma_A(x) = \sigma_B(x)$.*

Proof. Assume that $x \in A$ is normal. Then let:

$$A(x) := \overline{\{p(x, x^*) : p \text{ is a polynomial over } \mathbb{C} \text{ in 2-variables } \}}$$

where we take norm closure in A. This is called the *C^*-subalgebra generated by x*, in particular $A(x)$ is a commutative, unital, C^*-subalgebra of A. Then we have by Theorem 4.10, $\sigma_{A(x)}(x) = \{\varphi(x) : \varphi \in \Phi_{A(x)}\}$. As $A(x) \subseteq A$ we have $\sigma_A(x) \subseteq \sigma_{A(x)}(x)$. By Corollary 4.16 we have that

$$\{\varphi(x) : \varphi \in \Phi_{A(x)}\} \subset \begin{cases} \mathbb{R} & \text{if } x \text{ is self-adjoint} \\ S^1 & \text{if } x \text{ is unitary.} \end{cases}$$

Hence (1) and (2) is proven.

For (3), we firstly assume that x is self-adjoint. Then as $\sigma(x) \subseteq \mathbb{R}$, we have $\sigma(x) = \partial\sigma(x)$. We also have $\partial_B(x) \subseteq \partial_A(x)$ (why?) so

$$\sigma_B(x) = \partial\sigma_B(x) \subseteq \partial\sigma_A(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x).$$

Hence we have $\sigma_A(x) = \sigma_B(x)$. Now suppose x is normal, we have for any $y \in A$, $y \in G(A)$ if and only if $y^* \in G(A)$. As a result

$$\begin{aligned}\lambda - x \in G(A) &\Leftrightarrow \lambda - x, \bar{\lambda} - x^* \in G(A)\\ &\Leftrightarrow \eta := \left(\bar{\lambda} - x^*\right)(\lambda - x) \in G(A)\\ &\Leftrightarrow \eta \in G(B) \quad \text{(as } \eta \text{ is self-adjoint)}\\ &\Leftrightarrow \lambda - x, \bar{\lambda} - x^* \in G(B)\\ &\Leftrightarrow \lambda - x \in G(B)\end{aligned}$$

where we have used the fact that, two commutative elements x, y is invertible if and only if xy is invertible (why?). □

We also have the square root of a positive element of A:

Definition 4.18. *$x \in A$ is positive if x is self-adjoint with non-negative spectrums. In this case we write $x \geq 0$.*

Proposition 4.19. *Let A be an unital C^*-algebra (not necessarily commutative). If x is positive, then there exists a unique positive $y \in A$ such that $y^2 = x$, which is called the* positive square root *of x. We write $y = x^{1/2}$.*

Proof. For existence, let B be any commutative, unital C^*-subalgebra of A such that $x \in B$. By the Commutative Gelfand-Naimark Theorem, there exists a compact Hausdorff space K and $\theta : C(K) \to B$ which is an isometric $*$-isomorphism. Let $f = \theta^{-1}(x)$, which is a function on K. Then since x is self-adjoint so is f, and we have

$$f(K) = \sigma_{C(K)}(f) = \sigma_B(x) = \sigma_A(x) \subset [0, \infty),$$

hence f has an unique positive square root $\sqrt{f} := g$. Let $y = \theta(g)$. Then since g is $\mathbb{R}$-valued, it is self-adjoint and so is y. Moreover again we have:

$$\sigma_A(y) = \sigma_{C(K)}(g) = \sqrt{f(K)} \subset [0, \infty).$$

So y is positive. Also, $y^2 = \theta\left(g^2\right) = \theta(f) = x$, we have existence.

For the uniqueness, if $z \in B$, $z \geq 0$ such that $z^2 = x$, let $h = \theta^{-1}(z)$. Then $h \geq 0$ and $h^2 = f = g^2$. So by uniqueness of positive square root in $\mathbb{R}$, $h = g$. Hence $y = z$, and the uniqueness in B is obtained.

Consider the general case in A. If $y, z \in A$ are positive and $y^2 = z^2 = x$, then $yx = y^3 = xy$ and $zx = z^3 = xz$. Set

$$B_1 = \overline{\{p\,(x, x^*, y, y^*) : p \text{ is a polynomial in 4 variables}\}},$$

and

$$B_2 = \overline{\{p\,(x, x^*, z, z^*) : p \text{ is a polynomial in 4 variables}\}},$$

which is the C^*-subalgebra of A generated by x, y and the C^*-subalgebra of A generated by x, z, respectively. B_1 and B_2 are both unital and commutative.

Let $B = B_1 \cap B_2$, which is a commutative, unital C^*-subalgebra that contains x. Hence there exists an unique square root k of x in B. By uniqueness in B_1 and B_2 we get $y = k = z$. So we have uniqueness in A. □

Theorem 4.20. *(Polar Decomposition of Invertible Operators) Let $T \in \mathcal{B}(H)$ be invertible. Then there exist an unique unitary operator U and an unique positive operator R such that $T = RU$.*

Proof. Notice that TT^* is always positive and we define $R = \sqrt{TT^*}$. As T is invertible, $R^2 = TT^*$ is invertible so R is invertible. Set $U = R^{-1}T$, then we have

$$U^*U = T^*R^{-1}R^{-1}T = T^*(T^*)^{-1}T^{-1}T^* = I.$$

So existence follows. For uniqueness, let $T = RU$, then $TT^* = R^2$ so by uniqueness of square root, R is unique and hence U. □

4.3. **Holomorphic Functional Calculus.** Let $p(x)$ be a polynomial on $\mathbb{C}$. It is known when $A = \mathcal{B}(X)$ for some Banach space X, we have the spectrum mapping theorem:

Theorem 4.21. *(Spectrum Mapping Theorem) For every $T \in \mathcal{B}(X)$ we have $\sigma(p(T)) = p(\sigma(T))$. More generally, for any unital Banach algebra A and $x \in A$, we have $\sigma(p(x)) = p(\sigma(x))$.*

Proof. See Oxford Functional Analysis II note at `https://courses.maths.ox.ac.uk/course/view.php?id=5548`, Theorem 5.7. □

This theorem seems intuitive in every sense and we can seek its generalization. For example, we already know that

$$\exp(T) := \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

converges for every $T \in \mathcal{B}(X)$. Is it necessarily true that, $\exp(\sigma(T)) = \sigma(\exp(T))$? Can we define $f(T)$ for other, say, holomorphic functions f? Is it necessarily true that, $f(\sigma(T)) = \sigma(f(T))$ then?

Let U be an open subset, we notice from Cauchy Integral formula, for any f holomorphic on U, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w$$

where γ lies in U and z lies in the inside of γ. It is intuitive that, if we change z into T we can define $f(T)$ via such "operator-valued" Cauchy integral formulas. But we need some formal setup here.

Proposition 4.22. *Let X be a Banach space and $f : [a, b] \to X$ be a continuous function. Take any sequence of partition $\pi^n = \{t_i^n : a = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = b\}$ with vanishing mesh. Then the limit of Riemann sum*

$$\lim_{n\to\infty} \sum_{k=1}^{N(\pi^n)} f(t_{k-1}^n)(t_k^n - t_{k-1}^n) =: \int_a^b f(t) \, \mathrm{d}t \in X$$

is independent of $(\pi^n)_{n\geq 1}$. We call it the integral of f on $[a, b]$.

It is routine to check this proposition.

Lemma 4.23. *For any $\phi \in X^*$, we have*

$$\phi\left(\int_a^b f(t) \, \mathrm{d}t\right) = \int_a^b (\phi \circ f)(t) \, \mathrm{d}t.$$

Exercise 4.24. *Prove that*

$$\left\| \int_a^b f(t) \, \mathrm{d}t \right\| \leq \int_a^b \|f(t)\| \, \mathrm{d}t.$$

We need to give the notion of holomorphic function taking values inside Banach spaces.

Definition 4.25. *Let $U \subset \mathbb{C}$ be a domain, X be a normed space, and $f : U \to X$ be a function. f is said to be* holomorphic *if the limit*

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists in X for any $z \in U$.

We immediately have

Lemma 4.26. *Let f be holomorphic on U, then for any $\phi \in X^*$, $\phi \circ f : U \to \mathbb{C}$ is a holomorphic function in usual sense.*

Proof. By continuity of ϕ we have

$$\lim_{w \to z} \frac{(\phi \circ f)(w) - (\phi \circ f)(z)}{w - z} = \lim_{w \to z} \phi \left(\frac{f(w) - f(z)}{w - z} \right) = \phi \left(\lim_{w \to z} \frac{f(w) - f(z)}{w - z} \right)$$

exists in $\mathbb{C}$. □

Lemma 4.27. *Let f be a bounded holomorphic function on $\mathbb{C}$ (i.e. an entire function). Then f is constant*

Proof. For any $\phi \in X^*$ we must have $\phi \circ f$ is bounded entire and hence constant. Take any $z \in \mathbb{C}$, we have $\phi(f(z)) = \phi(f(0))$ for any $\phi \in X^*$. By a consequence of Hahn-Banach theorem we see that $f(z) = f(0)$. □

Definition 4.28. *Let $\gamma : [a, b] \to \mathbb{C}$ be a path (i.e. continuously differentiable map). For any continuous function $f : \mathrm{Im}(\gamma) \to \mathbb{C}$, we define*

$$\int_\gamma f(z) \, \mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t.$$

We have similar properties as in complex integrations, in particular the Cauchy theorem holds.

Theorem 4.29. *(Cauchy Theorem) Let U be a domain of $\mathbb{C}$ and γ be a closed path in U such that the inside of γ all lies in U. Then*

$$\int_\gamma f(z) \, \mathrm{d}z = 0.$$

Proof. For any $\phi \in X^*$ we have

$$\phi \left(\int_\gamma f(z) \, \mathrm{d}z \right) = \int_\gamma \phi(f(z)) \, \mathrm{d}z = 0$$

by the usual Cauchy theorem. By Hahn-Banach $\int_\gamma f(z) \, \mathrm{d}z = 0$. □

For any rational function $r = p/q$ for polynomials p and q and $x \in A$ we define $r(x) = p(x)(q(x))^{-1}$ whenever $q(x)$ is invertible. We now apply the Cauchy's integral formula:

Proposition 4.30. *Let A be a commutative, unital Banach algebra, $U \subset \mathbb{C}$ be a domain and $x \in A$. Let $\mathscr{O}(U)$ be the space of holomorphic functions on U. Suppose $K = \sigma(x) \subset U$. Let γ be a closed path in $U \backslash K$ such that:*

$$I(\Gamma, \omega) = \begin{cases} 0 & \text{if } \omega \notin U \\ 1 & \text{if } \omega \in K \end{cases}$$

where I is the winding number. Define $\Theta_x : \mathscr{O}(U) \to A$ by

$$\Theta_x(f) := \frac{1}{2\pi i} \int_\gamma f(z)(z-x)^{-1} \, \mathrm{d}z.$$

Then:

1. *Θ_x is well-defined, linear and continuous, where $\mathscr{O}(U)$ is equipped with the supremum norm $\|f\|_\gamma = \sup_{z \in \gamma} |f(z)|$;*
2. *$\Theta_x(r) = r(x)$ for a rational function r without poles in U;*
3. *For any $\varphi \in \Phi_A$, we have $\varphi(\Theta_x(f)) = f(\varphi(x))$, and*

$$\sigma\left(\Theta_x(f)\right) = f\left(\sigma(x)\right) = \left\{f(\lambda) : \lambda \in \sigma(x)\right\}.$$

Proof. (1) As $z - x$ is invertible for any $z \in \gamma$, $z \to (z-x)^{-1}$ is continuous, we see Θ_x is well-defined. Linearity follows easily. As $z \to \|(z-x)^{-1}\|$ is also continuous we see that it is bounded (say by C_γ) on γ, and hence

$$\|\Theta_x(f)\| \leq \frac{C_\gamma}{2\pi} \|f\|_\gamma \ell(\gamma),$$

continuity follows.

For (2), we check that $\Theta_x(\mathbf{1}_U) = 1 = \mathbf{1}_U(x)$ first:

$$\Theta_x(\mathbf{1}_U) = \frac{1}{2\pi i} \int_\gamma \mathbf{1}_{z \in U} (z-x)^{-1} \, \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma (z-x)^{-1} \, \mathrm{d}z.$$

By Cauchy theorem and decomposing contour, the integral on γ agrees with the integral on $|z| = R$ for some $R > \|x\|$, so we have

$$\begin{aligned} \Theta_x(\mathbf{1}_U) &= \frac{1}{2\pi i} \int_{|z|=R} (z-x)^{-1} \, \mathrm{d}z = \frac{1}{2\pi i} \int_{|z|=R} \sum_{k=0}^\infty \frac{x^k}{z^{k+1}} \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \sum_{k=0}^\infty x^k \int_{|z|=R} \frac{1}{z^{k+1}} \, \mathrm{d}z = \frac{1}{2\pi i} \sum_{k=0}^\infty 2\pi i \delta_{0,k} x^k = x^0 = 1. \end{aligned}$$

Then for rational functions $r \in \mathscr{O}(U)$, we can write $r(z) = p(z)/q(z)$, where p, q are polynomials with q having no zeros in U. Hence we have: $0 \notin \{q(\lambda) : \lambda \in \sigma(x)\} = q\left(\sigma(x)\right) = \sigma(q(x))$ by the classical spectrum mapping theorem.

As $0 \notin \sigma(q(x))$, $q(x)$ is invertible for all $x \in A$, we can define $r(x) = p(x)q(x)^{-1}$.

We only need to check $r(x) = \Theta_x(r)$. Notice that:

$$r(z) - r(x) = (p(z)q(x) - q(z)p(x))q(z)^{-1}q(x)^{-1} = (z-x) \underbrace{s(z,x)q(z)^{-1}q(x)^{-1}}_{\text{analytic in } z \text{ on } U}.$$

The factorization holds as $p(z)q(x) - q(z)p(x) = 0$ for $x = z$. Hence

$$\begin{aligned}\Theta_x(r) &= \frac{1}{2\pi i}\int_\gamma (r(z) - r(x) + r(x))(z-x)^{-1}\ \mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\gamma s(z,x)q(z)^{-1}q(x)^{-1}\ \mathrm{d}z + r(x)\times\frac{1}{2\pi i}\int_\gamma (z-x)^{-1}\ \mathrm{d}z \\ &= r(x).\end{aligned}$$

The first integral is 0 by Cauchy theorem and the second one is 1 as $\Theta_x(\mathbf{1}_U) = 1$.

For (3) we firstly notice that for any $\varphi \in \Phi_A$ we have $\varphi((z-x)^{-1}) = (z-\varphi(x))^{-1}$. By the usual Cauchy integral formula we have

$$\varphi\left(\Theta_x(f)\right) = \frac{1}{2\pi i}\int_\gamma f(z)(z-\varphi(x))^{-1}\ \mathrm{d}z = f(\varphi(x)).$$

Hence we have

$$\begin{aligned}\sigma\left(\Theta_x(f)\right) &= \{\varphi\left(\Theta_x(f)\right) : \varphi \in \Phi_A\} = \{f(\varphi(x)) : \varphi \in \Phi_A\} \\ &= \{f(\lambda) : \lambda \in \sigma(x)\} = f(\sigma(x)).\end{aligned}$$

□

Now we can state the holomorphic functional calculus:

Theorem 4.31. *(Holomorphic Functional Calculus) Let A be a commutative, unital Banach algebra. Let $x \in A$ and let $U \subset \mathbb{C}$ be a domain with $\sigma_A(x) \subset U$. There exists an unique unital, continuous homomorphism $\Theta_x : \mathscr{O}(U) \to A$ such that $\Theta_x(\mathrm{id}_U) = x$. Moreover, $\varphi\left(\Theta_x(f)\right) = f(\varphi(x))$ for all $\varphi \in \Phi_A$ and $f \in \mathscr{O}(U)$, and we have:*

$$\sigma\left(\Theta_x(f)\right) = \{f(\lambda) : \lambda \in \sigma(x)\} = f\left(\sigma(x)\right).$$

Proof. We take the following lemma as granted:

Lemma 4.32. *(Runge's Approximation Theorem). Let $K \subset \mathbb{C}$ with $K \neq \emptyset$ be compact. Then for any f analytic on some open neighbourhood of K, and $\varepsilon > 0$, there exists a rational function r without poles in K such that $\|f - r\|_K := \sup_{z\in K}|f(z) - r(z)| < \varepsilon$.*

Consider all setups in Proposition 4.30. We just check Θ_x is a homomorphism: $\Theta_x(fg) = fg(x) = f(x)g(x) = \Theta_x(f)\Theta_x(g)$ for any rational functions f, g without pole in U. For general f, g we approximate f and g by rational functions and by continuity of Θ_x we are done.

We now check the uniqueness. Suppose there is another Φ_x satisfies all properties of Θ_x. Then $\Phi_x(p) = p(x)$ for all polynomials p, since if $p(z) = \sum_{k=0}^n a_k z^k$ we have

$$\Phi_x(p) = \sum_{k=0}^n a_k\Phi_x(z^k) = \sum_{k=0}^n a_k(\Phi_x(z))^k = \sum_{k=0}^n a_k x^k = p(x).$$

If p has no roots in U, then $0 \notin \sigma(p(x)) = \{p(\lambda) : \lambda \in K\}$, and so

$$p\Phi_x(1/p) = \Phi_x(p)\Phi_x(1/p) = \Phi_x(p\cdot(1/p)) = \Phi_x(1) = 1,$$

thus $\Phi_x(1/p) = p(x)^{-1}$. Therefore, we have $\Phi_x(r) = r(x)$ for any rational function r without pole in U. Hence by continuity and Runge's approximation theorem, we have that $\Phi_x = \Theta_x$. Other properties hold by approximate using Runge's approximation theorem. □

4.4. **Spectral Theorems.** Now we are about to prove some spectral theorems. Firstly we consider the C^*-algebra case:

Theorem 4.33. *(Spectral Theorem for Commutative, Unital C^*-algebra). Let A be a commutative, unital, C^*-subalgebra of $\mathcal{B}(H)$. Let $K = \Phi_A$, equipped with the weak* topology. Then, there exists a unique resolution of the identity P of H over K, such that*

$$\int_K \hat{T} \, \mathrm{d}P = T \text{ for any } T \in A$$

where $\hat{T}$ is the Gelfand transform of T. More over, we have $S \in \mathcal{B}(H)$ commutes with every $T \in A$ if and only if S commutes with every $P(E)$ for $E \in \mathcal{B}_K$.

Proof. We take the following theorem as granted:

Lemma 4.34. *(Riesz representation theorem RRT) Let K be a compact, Hausdorff space. Then the dual space of $C(K)$ is the space of complex Borel measure over K, equipped with the total variation norm*

$$\|\mu\|_1 = \sup\left\{\sum_{k=1}^n |\mu(A_k)| : K = \bigcup_{k=1}^n A_k \text{ and } A_1, A_2, \cdots, A_n \text{ are measurable}\right\}.$$

The identification map is

$$\mu \to \left(f \to \int_K f \, \mathrm{d}\mu\right) \in X^*.$$

By Theorem 4.14 we see that $T \to \hat{T}$ is a isometric $*$-isomorphism. In particular consider the map $m_{x,y} : C(\Phi_A) \ni \hat{T} \to \langle Tx, y\rangle$ for fixed $x, y \in H$. Clearly $m_{x,y}$ is in dual of $C(\Phi_A)$ and hence we have

$$m_{x,y}(\hat{T}) = \int_K \hat{T} \, \mathrm{d}\mu_{x,y}$$

for some complex Borel measure $\mu_{x,y}$, by RRT. By considering the self-adjoint element of A and then decompose everything in A into $h+ik$ for self-adjoint element h and k, we can deduce that $\mu_{x,y} = \overline{\mu_{y,x}}$, compare the proof of the first lemma in the proof of Theorem 4.14. Also, we have by linearity,

$$\int_K \hat{T} \, \mathrm{d}\mu_{\lambda x+y,z} = \langle T(\lambda x + y), z\rangle = \lambda\langle Tx, z\rangle + \langle Ty, z\rangle = \lambda \int_K \hat{T} \, \mathrm{d}\mu_{x,z} + \int_K \hat{T} \, \mathrm{d}\mu_{y,z}$$

So we have $\mu_{\lambda x+y,z} = \lambda\mu_{x,z} + \mu_{y,z}$. Similarly we have $\mu_{x,\lambda y+z} = \overline{\lambda}\mu_{x,y} + \mu_{x,z}$. Hence, by Riesz representation theorem for Hilbert space, we have that there exists a map $\Psi : L^\infty(K) \to B(H)$ such that for any $f \in L^\infty(K)$,

$$\int_K f \, \mathrm{d}\mu_{x,y} = \langle \Psi(f)(x), y\rangle.$$

Notice that Φ is bounded with $\|\Phi(f)\| \leq \|f\|_\infty$. Notice that

$$\langle x, \Psi(\overline{f})(y)\rangle = \overline{\langle \Psi(\bar{f})(y), x\rangle} = \overline{\int_K \bar{f} \, \mathrm{d}\mu_{y,x}} = \int_K f \, \mathrm{d}\mu_{x,y} = \langle \Psi(f)(x), y\rangle$$

so we have $\Psi(f)^* = \Psi(\bar{f})$. Also we have

$$\langle \Psi(\hat{T})x, y\rangle = \int_K \hat{T} \, \mathrm{d}_{x,y} = \langle Tx, y\rangle$$

for $T \in A$ and hence $\Psi(\hat{T}) = T$. For $S, T \in A$, we have

$$\int_K \hat{S}\hat{T}\ \mathrm{d}\mu_{x,y} = \int_K \widehat{ST}\mathrm{d}\mu_{x,y} = \langle S(Tx), y\rangle = \int_K \hat{S}\ \mathrm{d}\mu_{Tx,y}$$

and by the uniqueness in the RRT, we have $\hat{T}\ \mathrm{d}\mu_{x,y} = \mathrm{d}\mu_{T(x),\mu}$. For $f \in L^\infty(K)$,

$$\begin{aligned}\int_K f\hat{T}\ \mathrm{d}\mu_{x,y} = \int_K f\ \mathrm{d}\mu_{T(x),y} &= \langle \Psi(f)(T(x)), y\rangle = \langle T(x), \Psi(f)^*(y)\rangle \\ &= \langle T(x), \Psi(\bar{f})(y)\rangle = \int_K \hat{T}\ \mathrm{d}\mu_{x,\Psi(\bar{f})(y)}.\end{aligned}$$

So by the uniqueness of the RRT again, we have $f\ \mathrm{d}\mu_{x,y} = \mathrm{d}\mu_{x,\Psi(\bar{f})(y)}$. Thus, for $f, g \in L^\infty(K)$, we have:

$$\begin{aligned}\langle \Psi(fg)(x), y\rangle &= \int_K fg\ \mathrm{d}\mu_{x,y} = \int_K g\ \mathrm{d}\mu_{x,\Psi(\bar{f})(y)} \\ &= \langle \Psi(g)(x), \Psi(\bar{f})(y)\rangle = \langle \Psi(f)\Psi(g)(x), y\rangle.\end{aligned}$$

Hence we have $\Psi(fg) = \Psi(f) \circ \Psi(g)$.

Above all, Ψ is a continuous, unital, $*$-homomorphism. Set $P(E) = \Psi(\mathbf{1}_E)$ for any $E \in \mathcal{B}_K$. It is easy to check that P is a resolution of the identity of H over K. We finally check that, for $T \in A$, we have:

$$\int_K \hat{T}\ \mathrm{d}P_{x,y} = \int_K \hat{T}\ \mathrm{d}\mu_{x,y} = \langle Tx, y\rangle.$$

By (2) of Proposition 3.4 we see that

$$\int_K \hat{T}\ \mathrm{d}P = T$$

So we have the existence.

For the uniqueness, suppose $\int_K \hat{T}\ \mathrm{d}Q = T$. Then $\int_K \hat{T}\ \mathrm{d}Q_{x,y} = \langle Tx, y\rangle$, and then $Q_{x,y} = P_{x,y}$ by uniqueness of RRT. As $\langle Q(E)x, y\rangle = Q_{x,y}(E) = P_{x,y}(E)$ the uniqueness follows.

For the last part we have we have

$$\begin{aligned}&\langle (ST)x, y\rangle = \langle T(x), S^*y\rangle = \int_K \hat{T}\ \mathrm{d}P_{x,S^*y} \\ &\langle (TS)x, y\rangle = \int_K \hat{T}\ \mathrm{d}P_{Sx,y} \\ &\langle (S \circ P(E))x, y\rangle = \langle P(E)x, S^*(y)\rangle = P_{x,S^*y}(E) \\ &\langle (P(E) \circ S)x, y\rangle = P_{Sx,y}(E).\end{aligned}$$

So we have $ST = TS$ for all $T \in A$ if and only if $P_{x,S^*y} = P_{Sx,y}$ for all $x, y \in H$, if and only if $P_{Sx,y}(E) = P_{x,S^*y}(E)$ for any $E \in \mathcal{B}_K$, if and only if $S \circ P(E) = P(E) \circ S$. □

We also have the one for normal operators.

Theorem 4.35. *(Spectral Theorem for Normal Operators). Let $T \in \mathcal{B}(H)$ be normal. Then, there exists an unique resolution of the identity P of H over $K := \sigma(T)$ such that*

$$T = \int_{\sigma(T)} \lambda\ \mathrm{d}P.$$

Moreover, for $S \in \mathcal{B}(H)$, $ST = TS$ for any $T \in \mathcal{B}(H)$ if and only if $S \circ P(E) = P(E) \circ S$ for any $E \in \mathcal{B}_K$.

Proof. Consider $A(T) := A$ be the C^*-subalgebra generated by T. As T is normal we have, by Proposition 4.17 (3), that $\sigma(T) = \sigma_A(T)$. Then for $\varphi \in \Phi_A$ we have

$$\varphi\left(p\left(T, T^*\right)\right) = p\left(\varphi(T), \varphi\left(T^*\right)\right) = p(\varphi(T), \overline{\varphi(T)})$$

So φ is uniquely determined by $\varphi(T)$ on A. Hence the Gelfand transform $\hat{T} : \Phi_A \to \sigma_A(T)$ is injective. By Theorem 4.10 the map is surjective and continuous, and hence a homeomorphism (A continuous bijection between compact Hausdorff spaces is a homeomorphism). Hence we may apply Theorem 4.33 with $K = \sigma_A(T) = \sigma(T)$ to get the existence.

For uniqueness, assume that we have $T = \int_{\sigma(T)} \lambda \, \mathrm{d}Q$, then for any polynomial p,

$$p\left(T, T^*\right) = \int_K p(\lambda, \bar{\lambda}) \mathrm{d}Q$$

and hence $\int_K p(\lambda, \bar{\lambda}) \mathrm{d}Q_{x,y}$ is uniquely determined by Proposition 3.4. Now by Stone Weiestrass we approximate everything by p and arrive at the uniqueness of $\int_K f(\lambda) \mathrm{d}Q_{x,y}$ for any $f \in C(K)$. As a result, $Q_{x,y} = P_{x,y}$ so uniqueness follows. The moreover part is clear. □

We can now define a very useful version of Borel functional calculus.

Proposition 4.36. *(Borel Functional Calculus for Normal Operators) Let $T \in B(H)$ be a normal operator on H, and let $K = \sigma(T)$ be the spectrum of T, and let P be as in Theorem 4.35. Define:*

$$L^\infty(K) \to B(H) \quad \text{by} \quad f \mapsto f(T) \equiv \int_K f \, \mathrm{d}P$$

Then this map has the following properties:

1. *It is a unital, *-homomorphism, and $\mathrm{id}(T) = T$, where id is the identity map on $\mathbb{C}$ (i.e. $\int_K id \, \mathrm{d}P = id$ is the identity on H);*
2. *$\|f(T)\| \leq \|f\|_K$ for all $f \in L^\infty(K)$, with equality if $f \in C(K)$;*
3. *For $S \in B(H)$, we have $ST = TS$ if and only if $S \circ f(T) = f(T) \circ S$ for all $f \in L^\infty(K)$;*
4. *$\sigma(f(T)) \subseteq \overline{f(K)}$.*

We have essentially already proven this proposition. We now look at a few consequences:

Theorem 4.37. *(Polar Decomposition of Normal Operator) Suppose $T \in \mathcal{B}(H)$ is normal. Then there exists a unitary operator U and a positive operator R such that $T = RU$.*

Proof. Let $K = \sigma(T)$ and set

$$u(\lambda) = \begin{cases} \lambda / |\lambda| & \text{if } \lambda \in K \backslash \{0\} \\ 1 & \text{if } \lambda = 0 \in K. \end{cases}$$

Let $r(\lambda) = |\lambda|$. Then let $U = u(T)$ and $R = r(T)$. Then as $ru = \mathrm{id}_K$, we see that $RU = T$. □

Theorem 4.38. *(Representation of Unitary Operators) Let $U \in \mathcal{B}(H)$ be a unitary operator. Then, there exists a Hermitian operator Q such that $U = e^{iQ}$.*

Proof. Note that since U is unitary, we have $K := \sigma(U) \subset S^1$. By taking a suitable choice of logarithm, there exists $f : S^1 \to \mathbb{R}$ measurable and bounded such that $e^{if(t)} = t$ for any $t \in S^1$. Let $Q = f(U)$. Since $\sum_{n=0}^{N} \frac{(if(t))^n}{n!} \to t$ uniformly on S^1 we have

$$e^{iQ} := \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(iQ)^n}{n!} = \lim_{N\to\infty} \sum_{n=)}^{N} \frac{(if(U))^n}{n!} = U$$

Thus $U = e^{iQ}$. □