Hence

$$\phi'(R) = \begin{cases} 0 & R < a, \\ 2\pi G\left(\dfrac{a^2}{R^2} - 1\right) & a < R < b, \\ \dfrac{2\pi G\left(a^2 - b^2\right)}{R^2} & b < R. \end{cases}$$

If we integrate we have

$$\phi(R) = \begin{cases} A & R < a, \\ B - 2\pi G\left(\dfrac{a^2}{R} + R\right) & a < R < b, \\ \dfrac{2\pi G\left(b^2 - a^2\right)}{R} + C & b < R. \end{cases}$$

As $\phi(\infty) = 0$ then $C = 0$. As $\phi(b_-) = \phi(b_+)$ then

$$B - 2\pi G\left(\frac{a^2}{b} + b\right) = \frac{2\pi G\left(b^2 - a^2\right)}{b} \implies B = 4\pi G b.$$

Finally as $\phi(a_-) = \phi(a_+)$ then

$$A = 4\pi G b - 2\pi G\left(\frac{a^2}{a} + a\right) = 4\pi G\left(b - a\right),$$

and we have the same expressions for $\phi$ as were achieved by the previous method. ■