Final Honour School of Mathematics Part A

A4 Integration
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Do not turn this page until you are told that you may do so

1. Let m denote the Lebesgue measure on $\mathbb{R}$.

(a) [16 marks]

(i) Define the *Lebesgue outer measure* m^* and what it means for $E \subseteq \mathbb{R}$ to be *Lebesgue measurable.*

Solution:

$$m^*(E) = \inf\left\{\sum_{n=1}^{\infty} m(I_n) : I_n \text{ are intervals and } E \subseteq \bigcup_{n=1}^{\infty} I_n\right\},$$

where $m(I_n) = b_n - a_n$ when I_n has end points $a_n < b_n$ (details about how to assign lengths to intervals not required).
E is Lebesgue measurable iff

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

for all $A \subseteq \mathbb{R}$.
B 3

(ii) Show that for any subsets $E_1, E_2, \dots$ of $\mathbb{R}$,

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leqslant \sum_{n=1}^{\infty} m^*(E_n).$$

Solution: Let $\epsilon > 0$ and for each n find intervals $(I_{n,r})$ such that $E_n \subset \bigcup_{r=1}^{\infty} I_{r,n}$ and

$$\sum_{r=1}^{\infty} m(I_{n,r}) < m^*(E_n) + 2^{-n}\epsilon.$$

Then $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,r=1}^{\infty} I_{n,r}$ so that

$$m^*(\bigcup_{n=1}^{\infty} E_n) \leqslant \sum_{r,n=1}^{\infty} m(I_{n,r}) < \sum_{n=1}^{\infty} m^*(E_n) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $m^*(\bigcup_{n=1}^{\infty} E_n) \leqslant \sum_{n=1}^{\infty} m^*(E_n)$.
S 5 *This is an exercise and has appeared on relatively recent exams*

(iii) Let $F_1 \supseteq F_2 \supseteq \cdots$ be Lebesgue measurable sets with $m(F_1) < \infty$. Use countable additivity to show that

$$m\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} m(F_n).$$

Solution: Set $A_n = F_n \setminus F_{n+1}$ so that A_n are disjoint and $\bigcup_{n=1}^{\infty} A_n = F_1 \setminus \bigcap_{n=1}^{\infty} F_n$. Then by countable additivity

$$m(F_1 \setminus \bigcap_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty}(m(F_n \setminus F_{n+1})).$$

Also (finite) addivity gives $m(F_1) = m(F_1 \setminus \bigcap_{n=1}^\infty F_n) + m(\bigcap_{n=1}^\infty F_n)$ and $m(F_{n+1}) + m(F_n \setminus F_{n+1}) = m(F_n)$. So

$$\sum_{n=1}^{\infty}(m(F_n) - m(F_{n+1})) = m(F_1) - \lim_{n+1}^{\infty} m(F_n).$$

Since $m(F_1) < \infty$, the result follows.
S 5 This solution argues directly from countable additivity. I would expect many students to first prove the bookwork result $m(\bigcup_{n=1}^\infty E_n) = \lim m(E_n)$ when $E_1 \subseteq E_2 \subseteq \dots$ and then set $E_n = F_1 \setminus F_n$.

(iv) Let $(E_n)_{n=1}^\infty$ be a sequence of Lebesgue measurable sets with $\sum_{n=1}^\infty m(E_n) < \infty$ and set $E = \bigcap_{n=1}^\infty \bigcup_{r \geqslant n} E_r$. Show that $m(E) = 0$.

Solution: We note that $m(\bigcup_{n=1}^\infty E_n) \leqslant \sum_n m(E_n) < \infty$ (using (ii)). Hence we can use (iii) to obtain

$$m(E) = \lim_n m(\bigcup_{r \geqslant n} E_r) \leqslant \lim_n \sum_{r \geqslant n} m(E_r) = 0.$$

S 5 Unseen.

(b) [9 marks] Let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions from $[0,1]$ to $\mathbb{R}$.

(i) Show that for each $n \in \mathbb{N}$, there exists $a_n \in \mathbb{R}$ with

$$m\left(\{x \in [0,1] : |f_n(x)| > a_n/n\}\right) < 2^{-n}.$$

Solution: Fix $n \in \mathbb{N}$, and let $F_r = \{x \in [0,1] : |f_n(x)| > r/n\}$. Since $F_1 \supset F_2 \supset \dots$ and $\bigcap_r F_r = \emptyset$, there exists some r with $m(F_r) < 2^{-n}$ (by (a)(iii)). Take $a_n = r$.
N 4

(ii) Deduce that for almost all $x \in [0,1]$,

$$\frac{f_n(x)}{a_n} \to 0 \quad \text{as } n \to \infty.$$

Solution: Let $E_n = \{x \in [0,1] : |f_n(x)| > a_n/n\}$ so that $\sum_n m(E_n) < \infty$. Therefore for $E = \bigcap_{n=1}^\infty \bigcup_{m \geqslant n} E_m$, part (a)(iv) gives $m(E) = 0$. If $x \notin E$, then there exists n such that $x \notin \bigcup_{m \geqslant n} E_m$, i.e.

$$\left|\frac{f_m(x)}{a_m}\right| \leqslant \frac{1}{m} \text{ for all } m \geqslant n.$$

Therefore $f_m(x)/a_m \to 0$ as $m \to \infty$ for $x \notin E$.
N 5 Sourced from Stein and Shakarchi chapter 1.

2. (a) [5 marks] State *Fubini's* and *Tonelli's* theorems for a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.

Solution: Fubini: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable. Then, for almost all y, $x \mapsto f(x, y)$ is integrable, and if $F(y)$ is defined (for those y) by

$$F(y) = \int_{\mathbb{R}} f(x, y) dx,$$

then $F(y)$ is integrable and

$$\int_{\mathbb{R}^2} f(x, y) d(x, y) = \int_{\mathbb{R}} F(y) dy.$$

Similarly

$$\int_{\mathbb{R}^2} f(x, y) d(x, y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) dy \right) dx.$$

Tonelli: Such a function is integrable on $\mathbb{R}^2$ if either of the repeated integrals

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, y)| dx \right) dy \text{ or } \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, y)| dy \right) dx$$

is finite (and hence Fubini's theorem applies to $|f|$ and f).
B 5

(b) [15 marks] Let $a > 1$.

(i) Show that the function $f(x, y) = e^{-xy}$ is integrable on $[0, \infty) \times [1, a]$.

Solution: Note $e^{-xy} \geqslant 0$ on this range. For any $1 \leqslant y \leqslant a$,

$$\int_0^\infty e^{-xy} dx = \lim_{n \to \infty} \int_0^n e^{-xy} dx = \lim_{n \to \infty} \left(\frac{e^{-ny}}{y} - \frac{1}{y} \right) = \frac{1}{y}$$

using the monotone convergence theorem, and the fundamental theorem of calculus. Then

$$\int_1^a \int_0^\infty e^{-xy} dx dy = \log a$$

by the fundamental theorem of calculus. By Tonelli's theorem, e^{-xy} is integrable on the specified domain.
S 4 *Note that the calculation of the integral as* $\log a$ *really belongs to the next subpart.*

(ii) Show that

$$\int_0^\infty \frac{e^{-x} - e^{-ax}}{x} \mathrm{d}x = \log a.$$

Solution: By Tonelli's theorem

$$\log a = \int_0^\infty \int_1^a e^{-xy} dy dx = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x} dx.$$

where the second integral is calculated by the fundamental theorem of calculus.
S 3

(iii) Is the function $g(x,y) = e^{-xy} - ae^{-axy}$ integrable over $[0,1] \times [1,\infty)$? Justify your answer.

Solution: Firstly, for any $y \in [1,\infty)$

$$\int_0^1 (e^{-xy} - ae^{-axy})dx = \frac{e^{-ay} - e^{-y}}{y}$$

by the FTC. So

$$\int_1^\infty \int_0^1 (e^{-xy} - ae^{-axy})dxdy = \int_1^\infty \frac{e^{-ay} - e^{-y}}{y} dy$$

On the other hand, for $x \in (0,1)$,

$$\begin{aligned}\int_1^\infty (e^{-xy} - ae^{-axy})dy &= \lim_{n\to\infty} \int_1^n (e^{-xy} - ae^{-axy})dy \\ &= \lim_{n\to\infty} \left(\frac{-e^{-nx} + e^{-anx}}{x} - \frac{-e^{-x} + e^{-ax}}{x} \right) \\ &= \frac{e^{-x} - e^{-ax}}{x}\end{aligned}$$

by the MCT (applied separately to each term), and the FTC. Thus

$$\int_0^1 \int_1^\infty (e^{-xy} - ae^{-axy})dydx = \int_0^1 \frac{e^{-x} - e^{-ax}}{x} dx$$

If $g(x,y)$ is integrable on the specifed domain, then

$$\int_0^1 \frac{e^{-x} - e^{-ax}}{x} dx = \int_1^\infty \frac{e^{-ay} - e^{-y}}{y} dy$$

by Fubini's theorem. Replacing y by x in the right hand integral, this gives

$$\int_0^\infty \frac{e^{-x} - e^{-ax}}{x} dx = 0,$$

contradicting the previous part. Thus g is not integrable.
N 8. *This exercise is sourced from Priestley, where it's an omitted example.*

(c) [5 marks] Let $f\colon [0,1] \to \mathbb{R}$ be measurable. Suppose that the function

$$g(x,y) = f(x) - f(y)$$

is integrable on $[0,1] \times [0,1]$. Must f be integrable on $[0,1]$? Justify your answer.

Solution: By Fubini, for almost all y, $x \mapsto |f(x) - f(y)|$ is integrable. Fix such a y_0 so

$$\int_0^1 |f(x) - f(y_0)|dx < \infty.$$

Since for all $x \in [0,1]$, we have

$$|f(x)| \leqslant |f(x) - f(y_0)| + |f(y_0)|,$$

comparison gives $\int_0^1 |f(x)| < \infty$. Therefore f is integrable on $[0, 1]$.
N 5. *Not especially difficult, but different in style from problem sheets.*

3. (a) [9 marks]

(i) Define a *simple* function $\phi : \mathbb{R} \to [0, \infty]$, and state an expression for the Lebesgue integral of ϕ.

> **Solution:** A simple function is a measurable function which takes only finitely many values. Such a function ϕ can be written in the form $\phi = \sum_{i=1}^{n} \alpha_i \chi_{B_i}$ with $\alpha_i \geqslant 0$ and measureable sets B_i. Then $\int_{\mathbb{R}} \phi = \sum_{i=1}^{n} \alpha_i m(B_i)$.
> **B 2**. This appears as an opening question every so often, last in 2018.

(ii) Let $f : \mathbb{R} \to [0, \infty]$ be measurable. Explain how to define its Lebesgue integral $\int_{\mathbb{R}} f$.

> **Solution:**
> $$\int_{\mathbb{R}} f = \sup\{\int_{\mathbb{R}} \phi : 0 \leqslant \phi \leqslant f,\ \phi \text{ simple}\}.$$
> **B 2** as above.

(iii) Let $f : \mathbb{R} \to [-\infty, \infty]$ be measurable. What does it mean for f to be *Lebesgue integrable*, and in that case define $\int_{\mathbb{R}} f$.

> **Solution:** Let $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$. f is integrable iff $\int_{\mathbb{R}} f^+ < \infty$ and $\int_{\mathbb{R}} f^- < \infty$. In this case we define $\int_{\mathbb{R}} f = \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-$.
> **B 1** as above

(iv) Suppose that $f, g : \mathbb{R} \to [-\infty, \infty]$ are Lebesgue integrable with $f(x) \leqslant g(x)$ for all x. Show that $\int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} g$.

> **Solution:** Firstly assume that $0 \leqslant f \leqslant g$. If $\phi : \mathbb{R} \to [0, \infty]$ is simple and $\phi \leqslant f$, then $\phi \leqslant g$. Therefore $\sup\{\int_{\mathbb{R}} \phi : 0 \leqslant \phi \leqslant f,\ \phi \text{ simple}\} \leqslant \sup\{\int_{\mathbb{R}} \phi : 0 \leqslant \phi \leqslant g,\ \phi \text{ simple}\}$,and so
> $$\int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} g.$$
> Now in general, we have $0 \leqslant f^+ \leqslant g^+$ so $\int_{\mathbb{R}} f^+ \leqslant \int_{\mathbb{R}} g^+$ and $0 \leqslant g^- \leqslant f^-$ so $\int_{\mathbb{R}} g^- \leqslant \int_{\mathbb{R}} f^-$. Assembling this gives $\int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} g$.
> **S 4** This sort of property was described verbally in lectures but only the result was recorded.

(b) [9 marks] Determine whether the following measurable functions are Lebesgue integrable over the specified set. Justify your answers.

(i) $f(x) = \frac{\cos x}{x}$ over $(1, \infty)$.

> **Solution:** We have
> $$\int_{n\pi}^{(n+1)\pi} \left| \frac{\cos x}{x} \right| \geqslant \frac{1}{(n+1)\pi} \int_{n\pi}^{n+1} \pi |\cos x| = \frac{2}{(n+1)\pi}.$$
> Since $\sum_{n=1}^{\infty} \frac{2}{(n+1)\pi} = \infty$, f is not integrable over $[\pi, \infty)$, by the baby MCT (and so not integrable over $(1, \infty)$ either).
> **S 3** *The same example with* sin *rather than* cos *appears in the course.*

(ii) $g(x) = \frac{\cos(1/x)}{x}$ over $(0, 1)$.

Solution: We substitute using the monotonic bijection $h : (0,1) \to (1,\infty)$, $h(x) = 1/x$ which has continuous derivative $h'(x) = -x^{-2}$. Then as the f from the previous subpart is not integrable on $(1,\infty)$, it follows that $(f \circ h) \cdot h'$ is not integrable on $(0,1)$. But $f(h(x))h'(x) = -\frac{\cos(1/x)}{x} = -g(x)$. Thus g is not integrable.
S 3 We do similar substitutions on sheet 3.

(iii) $h(x) = \frac{\sin x}{e^x - 1}$ over $(0,\infty)$.

Solution: For $x \geqslant 1$, we have $\left|\frac{\sin x}{e^x-1}\right| \leqslant \frac{2}{e^x}$. We have $\int_1^n \frac{2}{e^x} = 2(1-e^{-n}) \to 2$. Therefore by the baby MCT, and comparison g is integrable on $[1,\infty)$. Noting that $\lim_{x\to 0} f(x) = 1$ by L'Hopital, if we define $g(0) = 1$, g restricts to a continuous function on $[0,1]$, which is therefore bounded and integrable on this domain. Hence g is integrable on $[0,\infty)$.
S 3

(c) [7 marks] Write $\mathcal{L}^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is integrable}\}$ and define $\|f\|_1 = \int_{\mathbb{R}} |f|$ for $f \in \mathcal{L}^1(\mathbb{R})$.

(i) Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be sequences in $\mathcal{L}^1(\mathbb{R})$, and suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are such that $f_n \to f$ and $g_n \to g$ almost everywhere, and that $g \in \mathcal{L}^1(\mathbb{R})$. Suppose that $|f_n| \leqslant g_n$ for all $n \in \mathbb{N}$ and $\|g_n\|_1 \to \|g\|_1$. Show $f \in \mathcal{L}^1(\mathbb{R})$ and $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$.

Solution: f is an a.e. limit of measurable functions so measurable. As $|f_n| \leqslant g_n$, we have $|f| \leqslant g$ a.e. and hence $f \in \mathcal{L}^1(\mathbb{R})$.
We have $g_n \pm f_n \geqslant 0$ and $g_n \pm f_n \to g \pm f$ a.e. Applying Fatou:

$$\int (g \pm f) \leqslant \liminf \int (g_n \pm f_n).$$

As $g_n \geqslant 0$, and $\|g_n\|_1 \to \|g\|_1$, we have $\int g_n \to \int g$. Therefore

$$\pm \int f \leqslant \liminf \int (\pm f_n)$$

i.e.

$$\int f \leqslant \liminf \int f_n \leqslant \limsup \int f_n \leqslant \int f$$

Therefore $\lim \int f_n = \int f$.
N 4 *Potentially hard, but based on proof of the DCT so reworked bookwork in form too. This could be made easier by adding a hint to use Fatou*

(ii) Show that $\|f_n - f\|_1 \to 0$.

Solution: We have

$$|f_n - f| \leqslant |f_n| + |f| \leqslant g_n + |f| \to g + |f|$$

almost everywhere. As $g + |f|$ is integrable, and $\|g_n + |f|\|_1 = \int(g_n + |f|) \to \int(g + |f|) = \|g + |f|\|_1$, we can apply the previous part to $|f_n - f| \to 0$ a.e. Therefore

$$\|f_n - f\|_1 = \lim \int |f_n - f| = 0,$$

as required.
N 3