The Surfaces of Delaunay

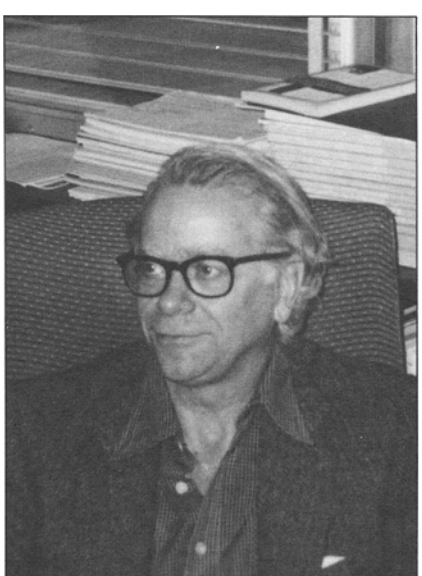
James Eells

1. Background

In 1841 the astronomer/mathematician C. Delaunay isolated a certain class of surfaces in Euclidean space, representations of which he described explicitly [1]. In an appendix to that paper M. Strum characterized Delaunay's surfaces variationally; indeed, as the solutions to an isoperimetric problem in the calculus of variations. That in turn revealed how those surfaces make their appearance in gas dynamics; soap bubbles and stems of plants provide simple examples. See Chapter V of the marvellous book [8] by D'Arcy Thompson for an essay on the occurrence and properties of such surfaces in nature.

More than 130 years later E. Calabi pointed out to me that the solutions to a certain pendulum problem of R. T. Smith [7] could be interpreted via the Gauss maps of Delaunay's surfaces [2]. And Eells and Lemaire [4] found that the Gauss map of one of those surfaces produces a solution to an existence problem in algebraic/differential topology.

The purpose of this article is to retrace those steps in an expository manner—as a revised version of [2].



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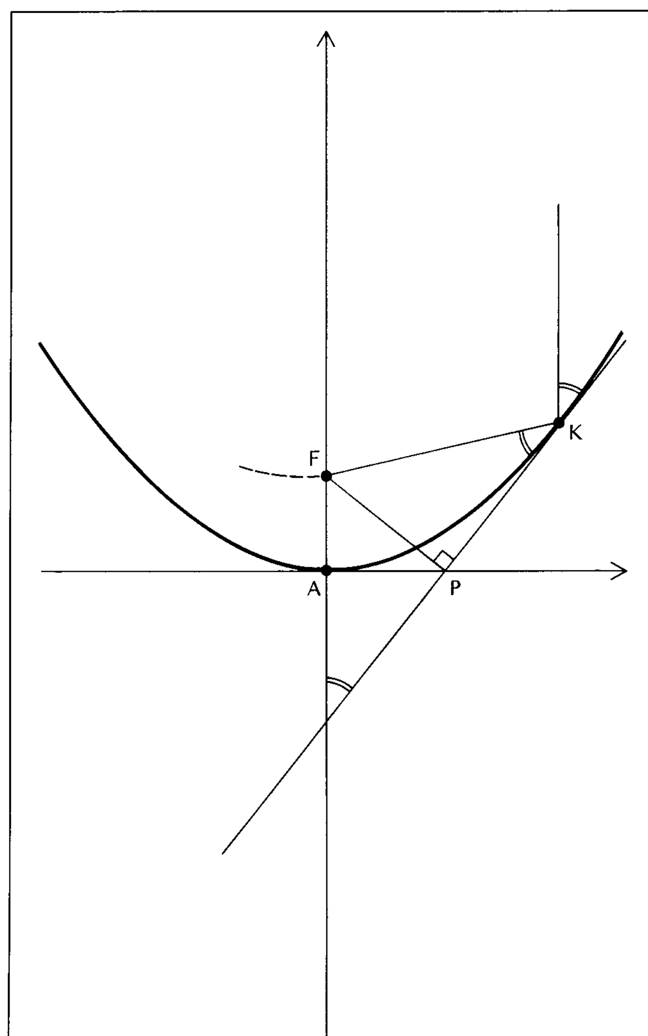
2. Roulettes of a Conic

The first step is to derive the equations describing the trace of a focus F of a nondegenerate conic ℓ as K rolls along a straight line in a plane. (Perhaps these derivations were better known a century ago!) We examine various cases separately.

ℓ IS A PARABOLA:



Here A is the vertex of ℓ. The line PK is tangent to ℓ at the point K. The following properties are elementary:

(1) Correspondingly marked angles are equal;
(2) FP is orthogonal to PK.

Thus we obtain

$$\overline{FA} = \overline{FP} \cos \angle AFP = \overline{FP} \cos \angle PFK.$$

Now we change our viewpoint and think of the tangent line PK as the axis—the x-axis—along which the parabola ℓ rolls. We denote the ordinate of F by y; and observe that

$$\cos \angle PFK = \frac{dx}{ds}$$

describes the rate of change of abscissa of F with respect to arc length s; i.e.,

$$\frac{dx}{ds} = \alpha,$$

where α denotes the angle made by the tangent with the x-axis. Thus setting $c = \overline{FA}$, we obtain the differential equation

$$c = y\frac{dx}{ds} = \frac{y}{\sqrt{1 + y'^2}}, \text{ or}$$

$$y' = \sqrt{\frac{y^2 - c^2}{c^2}}.$$

Its solution is the *catenary*

$$y = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right) = c \cosh x/c. \qquad \textbf{(2.1)}$$

That equation describes the shape of a flexible inextensible free-hanging cable—thereby explaining its name. In that context we can obtain the equation of the catenary as the Euler-Lagrange equation of the potential energy integral

$$P(y) = \int_{x_0}^{x_1} y \sqrt{\ + y'^2}\, dx,$$

subject to variations holding fixed the length integral

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx = L.$$

Indeed, from general principles we are asked to find a real number a and an extremal of the integral

$$J(y) = \int_{x_0}^{x_1} \left\{ \sqrt{1 + y'^2} + ay\sqrt{1 + y'^2} \right\} dx.$$

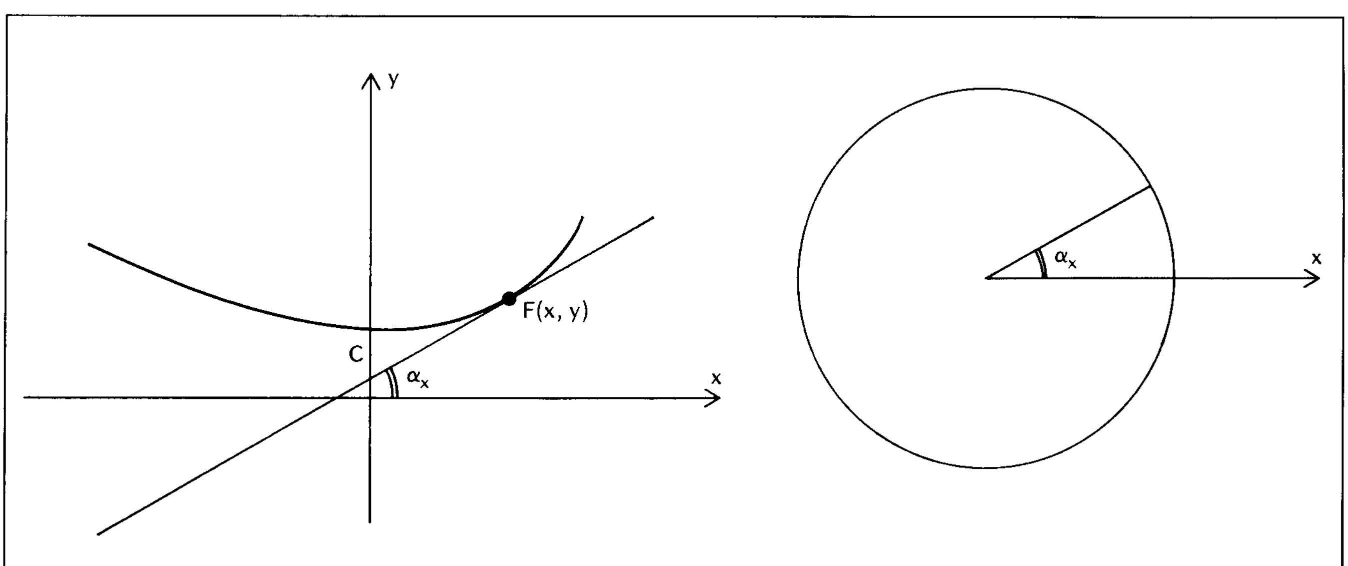
Its Euler-Lagrange equation has first integral

$$y' = \sqrt{\frac{(1 + ay)^2 - b^2}{b^2}} \text{ for } b \in \mathbf{R}.$$

The equation of the catenary is derived from this, choosing suitable normalizations.

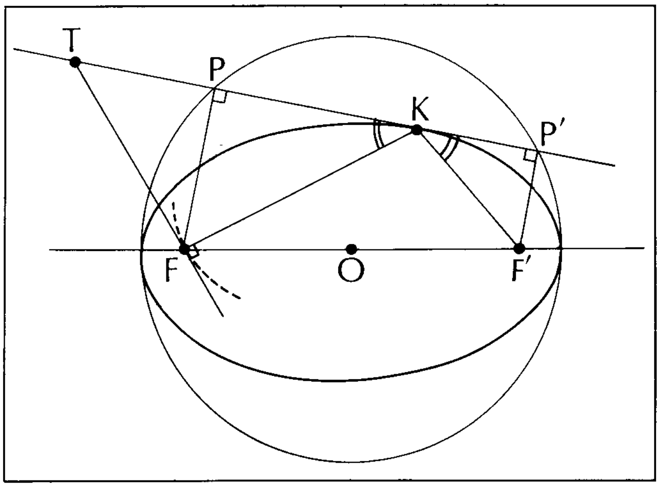
The curvature of ℓ is measured by the amount of turning of its tangent. That is expressed by the *Gauss map* of ℓ into the unit circle, given by $x \to \alpha_x$, where

$$\cos \alpha_x = \frac{dx}{ds} = \frac{c}{y}.$$



The Gauss map of the roulette of the parabola is injective onto an open semicircle.

ℓ IS AN ELLIPSE:



Here F and F' are the foci of ℓ; the 0 is its centre. The line PKP' is tangent to ℓ at K. Letting a and b denote the lengths of the semi-axes of ℓ, we obtain the following properties:

(1) $\overline{FK} + \overline{F'K} = 2a > 0$;
(2) the pedal equation $\overline{PF} \cdot \overline{P'F'} = b^2$ (see [9, Ch.VIII 6]);
(3) the normal to the locus of F passes through K.

Again using PK as x-axis,

$$\frac{y}{\overline{FK}} = \sin \angle FKP = \cos \angle FTP = \frac{dx}{ds}$$

$$\frac{y'}{\overline{F'K}} = \sin \angle F'KP' = \cos \angle F'T'P' = \frac{dx}{ds}.$$

From these we derive

$$y + y' = 2a\,\frac{dx}{ds},$$

$$y\,y' = b^2, \text{ so that}$$

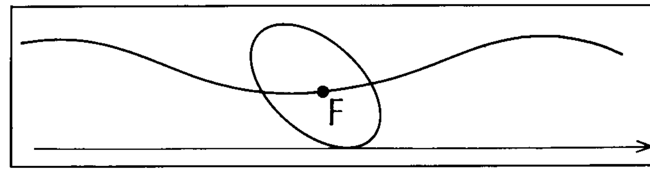
$$y^2 - 2ay\,\frac{dx}{ds} + b^2 = 0.$$

By analyzing all cases and taking $a \leq b$, we obtain

$$y^2 \pm 2ay\,\frac{dx}{ds} + b^2 = 0. \qquad \textbf{(2.2)}$$

The solutions to that differential equation can be given explicitly in terms of elliptic functions; see [1], [5, pp. 416–418].

The locus (of either focus) will be called the *undulary:*



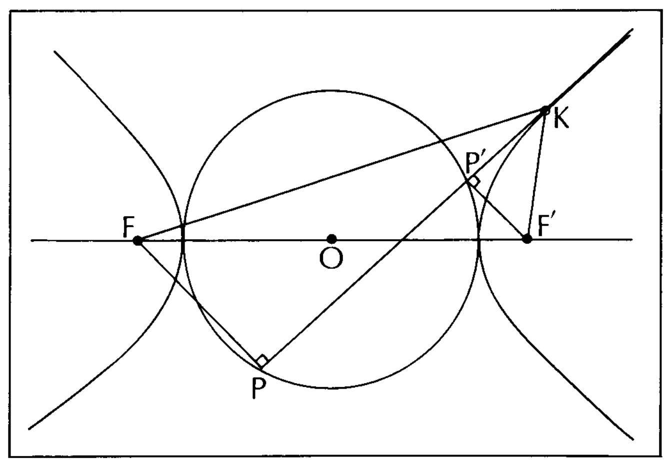
Its Gauss map is given by $x \to \alpha_x$, where

$$\cos \alpha_x = \mp \frac{y^2 + b^2}{2ay}.$$

It maps ℓ onto a closed arc of the unit circle.

There are two limiting cases, which are perhaps best handled separately: When $b \to a$ the undulary degenerates to a straight line, the locus of the centre of a circle rolling on a line. And where $b \to 0$ the undulary becomes a semicircle centred on the x-axis.

ℓ IS AN HYPERBOLA:



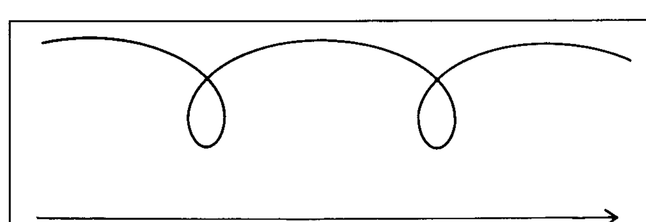
In analogy with the case of the ellipse, we have

(1) $\overline{FK} - \overline{F'K} = 2a > 0$;
(2) $\overline{PF} \cdot \overline{P'F'} = b^2$.

Thus we obtain the following differential equation for the locus of F, given as a first integral of an Euler-Lagrange equation:

$$y^2 \pm 2ay\,\frac{dx}{ds} - b^2 = 0. \qquad \textbf{(2.3)}$$

The loci of the two foci fit together to form the curve which we shall call the *nodary:*



Its Gauss map $x \to \alpha_x$ is governed by

$$\cos \alpha_x = \mp \frac{y^2 - b^2}{2ay}.$$

The Gauss map has no extreme points, and direct verification shows that it is surjective.

A *roulette of a conic* is a catenary, undulary, nodary, a straight line parallel to the x-axis, or a semicircle centred on the x-axis.

3. Surfaces of Revolution with Constant Mean Curvature

Rotating each of the roulettes about its axis of rolling produces five types of surfaces in Euclidean 3-space $\mathbf{R}^3$, called *the surfaces of Delaunay:* the *catenoids, unduloids, nodoids,* the *right circular cylinders,* and the *spheres.*

VARIATIONAL CHARACTERIZATION: We formulate the following isoperimetric principle, for the unduloid and nodoid (only minor technical changes being required for the other cases).

Consider graphs in $\mathbf{R}^2$ of non-negative functions

$$y: [x_0, x_1] \to \mathbf{R}(\geqslant 0)$$

with fixed volume of revolution

$$V(y) = \pi \int_{x_0}^{x_1} y^2 dx;$$

and extremize their lateral area

$$A(y) = 2\pi \int_{x_0}^{x_1} y^2 ds$$

holding the endpoints fixed. By general principles of constraint (under the heading of Lagrange's method of multipliers for isoperimetric problems [5]), we are led to the Euler-Lagrange equation associated with the integral

$$F(y) = \pi \int_{x_0}^{x_1} (y^2 dx + 2ay\, ds)$$

$$= \pi \int_{x_0}^{x_1} (y^2 + 2ay\sqrt{1 + y'^2}) dx.$$

Here a is a convenient real parameter. Its integrand f does not involve x explicitly, so we obtain a first integral from

$$0 = y'\left(f_y - \frac{d}{dx} f_{y'}\right) = \frac{d}{dx}(f - y' f_{y'}).$$

Thus $f - y' f_{y'} = \pm b^2$, where b is another real parameter. Consequently,

$$y^2 + \frac{2ay}{\sqrt{1 + y'^2}} \mp b^2 = 0.$$

But

$$\frac{1}{\sqrt{1 + y'^2}} = \frac{dx}{ds}$$

so the extremal equation for our variational problem coincides with that of the roulette of the ellipse or hyperbola ((2.2) and (2.3)).

GAUSS MAPS: In an analogy with the case of oriented curves in the plane (§2), we associate to any oriented surface M immersed in $\mathbf{R}^3$ its Gauss map $\gamma : M \to S$ (the unit 2-sphere centred at the origin in $\mathbf{R}^3$), defined by assigning to each point $x \in M$ the positive unit vector orthogonal to the oriented tangent plane to M at x. Its differential $d\gamma(x)$ can be interpreted as a symmetric bilinear form on the tangent space $T_x M$. Its eigenvalues λ_1, λ_2 are well determined up to order. The symmetric functions $K_x = \lambda_1 \lambda_2$ and $H_x = (\lambda_1 + \lambda_2)/2$ are called the *curvature* of M and the *mean curvature of the immersion* at x, respectively. For instance,

(1) the cylinder has $K \equiv 0$ and constant mean curvature $H \neq 0$;

(2) the sphere of radius R has constant curvature $K = 1/R^2$ and constant mean curvature $H = 1/R$;

(3) the catenoid has variable curvature K and mean curvature $H \equiv 0$;

(4,5) the unduloid and nodoid have variable curvature K and constant mean curvature $H \neq 0$.

These five surfaces were recognized by Plateau, using soap film experiments.

Say that a surface of constant mean curvature in $\mathbf{R}^3$ is *complete* if it is not part of a larger such surface. From Sturm's variational characterization, we obtain

DELAUNAY'S THEOREM: *The complete immersed surfaces of revolution in* $\mathbf{R}^3$ *with constant mean curvature are precisely those obtained by rotating about their axes the roulettes of the conics.*

Thus Delaunay's surfaces are those surfaces of revolution M in $\mathbf{R}^3$ which are maintained in equilibrium by the pressure of a field of force which acts everywhere orthogonally to M.

4. Harmonic Gauss Maps

An easy yet vitally important theorem of Ruh-Vilms [6] states that:

A surface M *immersed in* $\mathbf{R}^3$ *has constant mean curvature if and only if its Gauss map* γ: M $\to$ S *satisfies the equation*

$$\Delta\gamma = \|d\gamma\|^2\gamma,$$

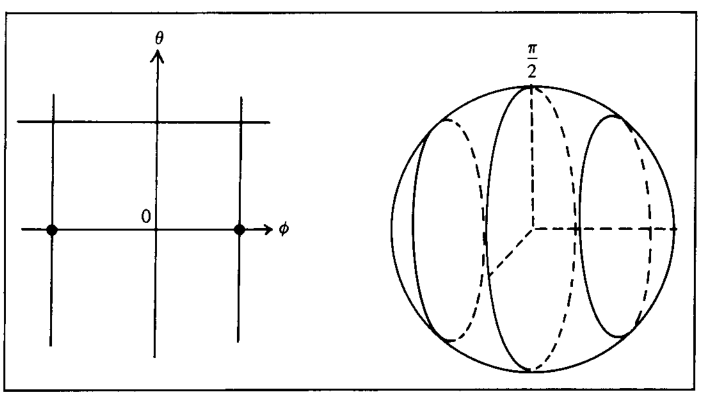
where Δ denotes the Laplacian of M with conformal structure induced from that of $\mathbf{R}^3$, and vertical bars the Euclidean norm at each point. Indeed, (4.1) *is the condition for harmonicity of the map* γ [3]—and is the Euler-Lagrange equation associated to the *energy* (or *action*) *integral*

$$E(\gamma) = \frac{1}{2}\int_M \|d\gamma\|^2.$$

E is a conformal invariant of M.

SMITH'S MECHANICS: Motivated by certain mechanical analogies, R. T. Smith [7] found solutions to equation (4.1) as maps $\gamma : \mathbf{R}^2 \to S$, as follows:

Think of points of $\mathbf{R}^2$ parametrized by angles (ϕ,θ), and use spherical coordinates on the sphere S:



If we restrict our attention to maps γ of the special form

$$(\phi,\theta) = (e^{i\theta}\sin\alpha(\phi), \cos\alpha(\phi)),$$

then the equation of harmonicity becomes the pendulum equation

$$\alpha'' = \frac{A}{2}\sin 2\alpha. \qquad \textbf{(4.3)}$$

We assume that $\alpha(0) = \pi/2$, so that the solution oscillates symmetrically about $\pi/2$.

Now a first integral of (4.3) is given by

$$\alpha' = \sqrt{\frac{C - A\cos^2}{2}}.$$

Again, that has an explicit solution in terms of elliptic functions. Furthermore, the *associated map* $\gamma : \mathbf{R}^2 \to$ S *is doubly periodic, factoring through the torus* T $= \mathbf{R}^2/\mathbf{Z}^2$ to produce a map $\gamma : T \to S$, as desired. Incidentally, the integrand of E is

$$\|d\gamma\|^2 = \alpha'^2 + \frac{A}{2}\sin^2\alpha.$$

Calabi made the beautiful observation that *Smith's maps* $\gamma : T \to S$ *are the Gauss maps of certain surfaces of Delaunay* [2].

A HARMONIC REPRESENTATIVE IN A HOMOTOPY CLASS: If we represent the torus T in the form $T = \mathbf{R}/a\mathbf{Z} \times \mathbf{R}/2\pi\mathbf{Z}$ and use polar coordinates (r,θ) on the unit sphere S, then a map from the cylinder to S of the form

$$r = \Phi(x),\ \theta = y$$

subject to the conditions $\Phi(0) = 0$, $\Phi(a) = \pi$ is harmonic if and only if Φ satisfies the pendulum equation (4.3) with $A = 1$. There are such solutions. Indeed [4], *the Gauss map of the nodoid induces a harmonic map of a Klein bottle* $\gamma : K \to S$. *Furthermore, that map is not deformable to a constant map.*

Hopf's classification theorem insures that the maps $K \to S$ are partitioned by homotopy into just two classes. Thus the harmonic map γ represents the nontrivial class.

References

1. C. Delaunay, *Sur la surface de révolution dont la courbure moyenne est constante.* J. Math. pures et appl. Sér. 1 (6) (1841), 309–320. With a note appended by M. Sturm.
2. J. Eells, *On the surfaces of Delaunay and their Gauss maps.* Proc. IV Int. Colloq. Diff. Geo. Santiago de Compostela (1978), 97–116.
3. J. Eells and L. Lemaire, *A report on harmonic maps.* Bull. London Math. Soc. 10 (1978), 1–68.
4. J. Eells and L. Lemaire, *On the construction of harmonic and holomorphic maps between surfaces.* Math. Ann. 252 (1980), 27–52.
5. A. R. Forsyth, *Calculus of variations.* Cambridge (1927).
6. E. A. Ruh and J. Vilms, *The tension field of the Gauss map.* Trans. Amer. Math. Soc. 149 (1970), 569–573.
7. R. T. Smith, *Harmonic mappings of spheres.* University of Warwick Thesis (1972).
8. D'A. W. Thompson, *Growth and form.* Cambridge (1917).
9. C. Zwikker, *The advanced geometry of plane curves and their applications.* Dover (1963).

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