Mathematical Entertainments

David Gale*

This column was originally called The Problem Corner. Under the previous editor the title was changed to Mathematical Entertainments, the idea being to broaden its content to include, for example, contests, historical notes, and the like. It is my intention, starting with this issue, to continue and even accelerate this trend. While problems and puzzles will still be welcome, there will also be emphasis on mathematical games, paradoxes, anecdotes, computer discoveries. In fact, the concept of an entertainment seems sufficiently vague to allow a wide variety of material, provided only that it should not require technical expertise in any particular area of mathematics. I hope readers of the Intelligencer *will find this sort of program congenial. Needless to say the success of the endeavor will depend crucially on getting good contributions from you, the readers, which are herewith eagerly solicited.*

The following theorem-joke was contributed by Hendrik Lenstra

"*Perfect squares don't exist.* Suppose that n is a perfect square. Look at the odd divisors of n. They all divide the largest of them, which is itself a square, say d^2. This shows that the odd divisors of n come in pairs a,b, where $a \cdot b = d^2$. Only d is paired to itself. Therefore the number of odd divisors of n is odd. This implies that the sum of all divisors of n is also odd. In particular, it is not $2n$. Hence n is not perfect, a contradiction: perfect squares don't exist."

Get it?

Remark: It seems the joke works only in English. In other languages a square is just a square (the theorem, however, is international).

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Computer-Generated Mysteries

The heading above describes a feature I would like to incorporate in these columns on a regular basis. Many mathematicians feel that the main impact of computers on mathematics has been not in solving problems, as one might have expected, but rather in posing them. The prime illustration is probably the recent activity in discrete dynamical systems stimulated by the celebrated computer experiments of Mitchell Feigenbaum. Perhaps explorations is a better description of this work, the appropriate analogy being not with physics or biology but with astronomy. The computer is the mathematician's telescope, which when used intelligently helps him/her to find out what is "out there" in the mathematical universe (this whole development should be a source of satisfaction to the Platonists who have been saying all along that, like stars and galaxies, mathematical phenomena are discovered, not invented).

Some quite recent work to be described in the next paragraphs gives another striking example of a set of phenomena that would probably never have been observed without the use of computers.

The Strange and Surprising Saga of the Somos Sequences

In investigating properties of elliptic theta functions, Michael Somos discovered an infinite sequence whose first 15 terms are

$$1, 1, 1, 1, 1, 1, 3, 5, 9, 23,$$
$$75, 421, 1103, 5047, 41783.$$

The sequence is defined by $a_i = 1$ for $0 \leq i \leq 5$ and

$$a_n = (a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2)/a_{n-6} \quad \text{for } n > 5. \tag{1}$$

The surprising fact was that the recursion generates integers as far as the eye = computer = telescope can see. In fact, for this example a telescope is not required. A good pair of binoculars will do. With a pocket scientific calculator one will easily, for example, verify that the next numerator above is divisible by 23 so that a_{15} is again an integer. What's going on?*

Upon seeing this phenomenon it occurred to a number of people to consider the simpler fourth-order recursion

$$a_n a_{n-4} = a_{n-1}a_{n-3} + a_{n-2}^2, \; a_0 = a_1 = a_2 = a_3 = 1 \tag{2}$$

Once again all entries turn out to be integers, but in this case the situation is manageable and several people have come up with proofs, the first one being given by Janice Malouf. We present here a variant due to George Bergman. First note that because of (2) every four consecutive terms of the sequence are pairwise relatively prime. For suppose this is true up to a_n. Then a_n would have a prime factor p in common with a_{n-1} or a_{n-3} if and only if p also divided a_{n-2}, contrary to the induction hypothesis.

We now show inductively that if $a_{n-4}, \ldots, a_n, \ldots, a_{n+3}$ are integers (clearly true for $n = 4$), then so is a_{n+4}, and hence all a_i. Writing $a_{n-3} = a$, $a_{n-2} = b$, $a_{n-1} = c$ we have $a_n a_{n-4} = ac + b^2$, so a_n divides $ac + b^2$. By the preceding paragraph we may apply (2) to the sequence modulo a_n giving

$$a, b, c, 0, \frac{c^2}{a}, \frac{c^3}{ab}, \frac{c^3}{a^2}, a_n a_{n+4} \equiv \frac{c^5}{a^3b^2}(ac + b^2) \equiv 0, \tag{3}$$

so a_n divides $a_n a_{n+4}$. □

Note that although the proof is very simple, it depends on the fortuitous fact that the factor $ac + b^2$ turns up on the fourth iteration. We will return to this point.

The same method works for the 5-term recursion

$$a_n a_{n-5} = a_{n-1}a_{n-4} + a_{n-2}a_{n-3}. \tag{4}$$

Actually in all of these recursions one may put arbitrary integers as coefficients of the terms $a_{n-i}a_{n-j}$ and still get integers, and this can be proved for the recursions (2) and (4).**

The next bit of progress came when Dean Hickerson proved that the original Somos sequence gives integers. In fact he showed something more general. Instead of starting with six one's he considered the sequence starting with indeterminates $a_0, a_1, \ldots, a_5$. The recursion then generates rational functions $a_n = p_n/q_n$ of these a_i and the theorem is that the denominators of these functions are always monomials with coefficient 1. This is of course clear for $a_0, \ldots, a_{11}$ but note that to compute a_{12} one must divide by $a_6 = (a_5a_1 + a_4a_2 + a_3^2)/a_0 = p_6/a_0$. One easily sees that hand computation of a_7, a_8, etc. quickly becomes unmanageable. This is of course what symbolic manipulation programs are designed for and using *Macsyma* Hickerson found that as in (3) above p_6 occurs as a factor of the numerator of a_{12} (which when reduced to lowest terms contains 194 terms!). Further *Macsyma* calculations are used to prove that p_6 is prime to $p_7, \ldots, p_{12}$ and an inductive argument is used to complete the proof. (Richard Stanley has also solved this problem using similar methods.)

But what have we learned? As Hickerson puts it, "The thing I dislike about my proof is that it doesn't explain why the result is true. It depends primarily on the fact that when you compute a_{12} there's an unexpected cancellation. But why does this happen?" Indeed the proof, rather than illuminating the phenomenon, makes it, if anything, more mysterious. I report this with some embarassment, since I have earlier asserted in this same journal that a proof in mathematics is in some sense equivalent to an explanation. We now see that this clearly need not be the case. Perhaps, if and when we find the "right" proof the situation will become clarified, but must there necessarily be a right proof? One is reminded of the proof of the four-color theorem? One of the most interesting features of the Somos problem, it seems to me, is that it leads to this sort of speculation.

Getting back to the question at hand, having found proofs for recursions of order 4, 5, 6, and empirical evidence for 7, it turns out that those of order 8 and above do not give integers. You will easily confirm with your pocket calculator, for example, that for the recursion of order 8, a_{17} is a fraction. Curiouser and curiouser.

The next discovery is due to Raphael Robinson, who found that the integer property of recursions (1), (2), and (4) was (apparently) shared by an infinite family of recursions. For any $k \geq 6$ start with k ones and then use the recursion

$$a_n a_{n-k} = a_{n-1}a_{n-k+1} + a_{n-2}a_{n-k+2} \quad \text{or} \tag{5}$$

$$a_n a_{n-k} = a_{n-1}a_{n-k+1} + a_{n-2}a_{n-k+2} + a_{n-3}a_{n-k+3}. \tag{5'}$$

The fact that one is now dealing with an infinite collection of sequences would seem to put the problem out of range of *Macsyma*-type proofs.

At this point my pocket calculator convinced me that for any $0 < \ell < m < k$ the recursion

$$a_n a_{n-k} = x a_{n-\ell} a_{n-k+\ell} + y a_{n-m} a_{n-k+m} \tag{6}$$

*Somos actually discovered his sequences eight years ago but did not succeed in capturing the attention of the mathematical community until the summer of 1989.

**If the integer coefficients are allowed to be negative, then it may happen that some $a_n = 0$, in which case we shall make the convention that the sequence terminates at that point.

gives integers, generalizing (5). Further investigations again by Robinson lead to the following

Conjecture: For any $p, q, r < k$ the recursion

$$a_n a_{n-k} = x a_{n-p} a_{n-k+p} + y a_{n-q} a_{n-k+q} + z a_{n-r} a_{n-k+r} \quad (7)$$

generates integers if and only if p, q, r can be chosen so that $p + q + r = k$.

(Robinson's evidence is only for the case $x = y = z = 1$. The arbitrary x, y, z are my responsibility.) This would subsume (5′) and (6). Namely (6) corresponds to choosing $p = \ell$, $q = k - m$, $r = m - \ell$, and $z = 0$ and (5′) corresponds to choosing $p = 1$, $q = 2$, $r = k - 3$, $x = y = z = 1$.

The story is not over. Dana Scott set up a program for the simplest case $k = 4$ but forgot to square the term a_{n-2}, yet the recursion still gave integers! In fact, it turns out that recursion (2) can be generalized to

$$a_n a_{n-4} = a_{n-1}^p a_{n-3}^q + a_{n-2}^r \quad \text{for any } p, q, r > 0 \quad (8)$$

and the Bergman proof goes through as it does for recursion (4) with arbitrary exponents. On the other hand, one cannot choose arbitrary exponents *and* coefficients. In fact, the recursion $a_n a_{n-4} = 2a_{n-1}a_{n-3} + a_{n-2}$ does not give integers (although if the right-hand side is $a_{n-1}a_{n-3} + y a_{n-2}$ it can be proved that the recursion gives integers for all y).

Recursion (8) is interesting, because in all the other examples the right-hand side was homogeneous. Was this a red herring? Perhaps, but when we go to three-term sequences, we can no longer throw in arbitrary exponents. In fact, if one forgets to square the term a_{n-3} in the original sequence (1), one gets fractions.

Perhaps the simplest recursion of all has been discovered by Scott. Namely, for any k

$$a_n a_{n-k} = a_{n-1}^2 + \ldots + a_{n-k+1}^2 \quad (9)$$

which seems to work for all k. Other "good" recursions seem to be

$$a_n a_{n-k} = a_n a_{n-2} + \ldots + a_{n-k+2} a_{n-k+1} \quad (10)$$

and for k odd

$$a_n a_{n-k} = a_{n-1} a_{n-2} + a_{n-3} a_{n-4} + \ldots + a_{n-k+2} a_{n-k+1}. \quad (11)$$

These recursions break new ground, since the right-hand side may have any number of terms, whereas in previous examples three terms seemed to be the maximum. For $k = 4$ the Bergman proof works for (9) and (11) but not for (10), which (for the moment) remains unsolved.

I don't want to drag this out indefinitely, for it seems new examples of Somos sequences are coming in faster than I can write them down. There is a whole area in which one uses recursions like (1) but starts with sequences other than all ones, e.g., ones and twos or ones and minus ones. Experiments indicate that sometimes one gets integers, other times not, but there seems to be no discernable pattern. On the positive side, using the ideas of Hickerson, Gale, and Robinson have proved integrality for the sequences (5) (but not (5′)). I strongly suspect that by the time this appears in print much more will be known about Somos sequences. Perhaps the problem will even have been solved, but as of this writing the situation remains intriguingly mysterious.

[*Added in Proof:* My suspicions seem to have been justified. During the month since the original manuscript was submitted Ben Lotto, using the ideas of Hickerson but no computer calculation, has shown that (9) always gives integers. The method doesn't seem to work however for (10) and (11), although Robinson, using an entirely different but elementary argument, has shown that (10) gives integers for $k = 4$, settling the question raised two paragraphs above. Also conjecture (7) has been proved for $k = 7$, $p = q = r = 1$ (using *Mathematica* rather than *Macsyma* this time). Finally, Robinson has discovered a whole set of periodicity phenomena which occur when the values of the terms in the sequences are reduced modulo n. Periodicity has been proved for (2), (4) and for (10) with $k = 4$ and (9) for $k = 3$, but remains unexplained (so far) otherwise.]

Problems

Derivatives eventually zero: Problem 91-1 by E. M. E. Wermuth (Jülich, Germany)

Let f be a C^∞ function defined on some open interval (a,b) such that for every x in (a,b) there is an integer $n(x)$ such that $f^{(n(x))}(x) = 0$. Show that f is a polynomial. (For multidimensional versions of the problem and its history see *MR*90e:26040.)

Solutions

Distinct digits: Problem 90-4 by S. H. Weintraub (Louisiana State University, USA)

Without using any aid to computation other than pen (or pencil) and paper, find an integer n with the property that the *leading* ten digits of $n^{10^{1000000}}$ are all distinct.

Solution by F. B. Strauss (Univ. of Texas at El Paso, USA)

Let $a = 1023456790$ and write m in place of $10^{1000000}$. The real number $a^{1/m}$ lies between 1 and 2 and its decimal representation can be given as $r + s$, where $r = \sum_{i=0}^{M} c_i 10^{-i}$, $s = \sum_{i=M+1}^{\infty} c_i 10^{-i}$, and M is a positive integer to be chosen to suit our purposes. We have

$$(r + s)^m = a = r^m + \sum_{k=0}^{m-1} \binom{m}{k} r^k s^{m-k}$$

and

$$s < 10^{-M}.$$

Hence r^m differs from a by

$$\sum_{k=0}^{m-1} \binom{m}{k} r^k s^{m-k} < 10^{-M} \sum_{k=0}^{m-1} \binom{m}{k} r^k$$

$$= 10^{-M}((r+1)^m - r^m) < 10^{-M}(3^m - 1)$$

$$< 10^{-M} 3^m.$$

Now choose M so large that $10^{-M}3^m < 1$ and r^m is an integer. The choice $M = m$ will accomplish this; then $a - 1 < r^m < a$. Finally, take $n = 10^m r$. Then $n^m = 10^{m^2} r^m$ has the same ten leading digits as r^m, namely, those of the number $a - 1 = 10234567889$.

A number of people have noted that the solution to Problem 89-7 by Andrew Lenard is incomplete. He concludes that a continuous strictly monotone function from **R** to **R** is a homeomorphism but fails to show it is surjective. A correct proof can be given by noting that the fixed points of f are a closed set and then applying Lenard's method to each of the countable sets of open intervals of non-fixed points.

Is there a mathematics gene?

The four-year-old niece of a mathematical logician was playing a game in which she was the conductor on a train and her mother was a passenger. "Wait a minute," said Nancy, "we have to get some paper to make tickets." "Oh", said her mother, who had probably had a long day, "Do we really need them? After all, it's only a pretend game with pretend tickets." "Oh no, mommy, you're wrong," replied Nancy, "They're pretend tickets, but it's a real game."

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