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The Bellows Conjecture

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Abstract. We show that any continuous flex that preserves the edge lengths of a closed triangulated surface of any genus in three-space must flex in such a way that the volume it bounds stays constant during the flex.

1. Introduction

Consider a triangulated polyhedral surface S in three-space. Regard the edges of S as rigid bars, and regard the bars as joined at ideal universal joints at the vertices of S. There are several examples when S is a mathematically exact (flexible) mechanism (see Connelly [4] for example). Indeed there are even simpler examples of such surfaces where S may intersect itself and have various singularities. The simplest non-trivial example of this sort is a self-intersecting surface due to R. Bricard [2], and there are many others, for example Connelly [5].

For each such orientable singular surface S, it is possible to define the notion of the (generalized, signed) volume bounded by S, $\mathrm{vol}(S)$. When S is a (triangulated) embedded surface, $|\mathrm{vol}(S)|$ is indeed the volume of the bounded domain with S as boundary.

Suppose S_t, $0 \leq t \leq 1$, represents a flex of the surface S so that $S_0 = S$. In Connelly [3] it was conjectured that $\mathrm{vol}(S_t)$ is constant. This was called the *Bellows Conjecture* in the sense that it stated that there is no exact mathematical bellows.

Here we describe a proof of the Bellows Conjecture for any triangulated orientable surface mapped into three-space. The ideas here were inspired by a proof by I. Sabitov in [7] in case the surface S is homeomorphic to a sphere. A proof for the general case of a

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manifold will appear in [8]. In fact, the proof here follows the same plan as in Sabitov [8]. The basic addition here is a way of streamlining a key induction step using the theory of places instead of resultants. This has the advantage that the proofs are simpler and easier to find, but the calculation of the polynomial in the main result is much less explicit.

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2. An algebraic reformulation

Although we are primarily interested in surfaces in Euclidean three-space, it is useful to think of things in the following way. Let $p_i = (x_i, y_i, z_i)$, $i = 1, \ldots$ be a finite number of *points*, where the coordinates x_i, y_i, z_i together are algebraically independent quantities and generate a field $K = \mathbb{Q}(x_1, y_1, z_1, x_2, y_2, z_2, \ldots)$. So each point p_i belongs to K^3, $i = 1, \ldots$.

Let M be a triangulation of an orientable combinatorial 2-dimensional manifold. In other words M is a 2-dimensional simplicial complex such that the triangles (2-simplices) adjacent to a given vertex form a cycle, and there is a consistent orientation to the triangles.

Let $M \to L^3$ be a map that associates to each vertex i of M the point $p_i \in L^3$, where L is any field of characteristic not $2, 3$. We regard this map as a *singular surface* S. We define the *(generalized) volume* of S as

$$\mathrm{vol}(S) = \frac{1}{6} \sum_{[i,j,k] \in M_+} \det[p_i, p_j, p_k],$$

where the sum is taken over all positively oriented triangles $[i, j, k]$ of M (denoted by M_+). (In our notation describing a matrix, we will treat vectors as columns).

Let L be any field that contains a ring R. Recall from algebra (see Lang [6]) that an element $x \in L$ is defined to be *integral* over R if there are elements $a_i \in R$, $i = 0, 1, \ldots, n-1$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

For any field L we say that the *square of the edge length* between p_i and p_j in L^N is $(p_i - p_j)^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \ldots$, where $p_i = (x_i, y_i, z_i, \ldots)$.

The following is the main result of this paper.

Theorem 1. *For any (singular) orientable surface S in L^3, where L is any field of characteristic not 2 or 3, $12\,\mathrm{vol}(S)$ is integral over R, the ring generated by the squares of the edge lengths of S.*

Remark. In order to prove Theorem 1, it is enough to consider only the case when the field L is the field K described above. We can simply *specialize* each independent coordinate x_i, y_i or z_i to be the desired quantity in the field L. This provides a (ring) homomorphism

$K \to L$, and the integral condition for vol(S) is preserved since, in the monic polynomial above, each of the coefficients, as well as the volume itself, is in turn a polynomial (over the integers) in the coordinates of the configuration.

Another advantage in taking the coordinates to be independent in the field K is that the Theorem above provides a single polynomial for that particular configuration. But then (using the homomorphism above) the same polynomial will be satisfied by the volume for any other configuration, as long as the combinatorial type of the underlying manifold remains the same, for any field. In particular, we can take L to be the real field.

Corollary 1. *If S_t is a flex of an orientable (singular) surface in $\mathbb{R}^3$, then* vol(S_t) *is constant.*

Proof. The integrality condition and the remark above insures that there is a fixed monic polynomial that is satisfied by the volume, and that the coefficients of this polynomial are functions of the squared edge lengths and only depend on the combinatorial type of the underlying manifold. Thus there are at most a finite number of values for vol(S_t), independent of t, $0 \le t \le 1$. Since vol(S_t) is continuous in t, it must be constant.

3. Places

In order to streamline our tests for integrality, we repeat here some basic facts from Lang [6]. Suppose L and F are fields. Let $\varphi : L \to F \cup \{\infty\}$ be a function such that for all $x, y \in L$,

(i) $\varphi(x+y) = \varphi(x) + \varphi(y)$
(ii) $\varphi(xy) = \varphi(x)\varphi(y)$ and
(iii) $\varphi(1) = 1$,

where it is understood that for $a \in F$ (called a *finite* a) $a \pm \infty = \infty \cdot \infty = \frac{1}{0} = \infty$, $\frac{a}{\infty} = 0$, and if $a \neq 0$, $a \cdot \infty = \infty$. (The expressions $\frac{0}{0}$, $\frac{\infty}{\infty}, 0 \cdot \infty, \infty \pm \infty$ are *not* defined, and it is also understood that (i) and (ii) only hold when the right hand side is defined.)

We call such a function a *place* for the field L. Our basic tool for integrality is the following from Lang [6], page 12.

Lemma 1. *An element x in a field L containing the ring R is integral over R if and only if every place defined on L that is finite on R is finite on x.*

Corollary 2. *Suppose that x, y are both integral elements in a field containing a ring R. Then $x + y$ and $x - y$ are integral over R as well.*

4. The Cayley-Menger determinant

We need an algebraic condition on the set of distances between pairs of points that are satisfied when they exist in L^N, where L is any field and $N = 3, 4$. In the following vol$[p_1, \dots, p_n] = \frac{1}{(n-1)!}\det[p_1, \dots, p_n]$ is the $(n-1)$-dimensional volume of the simplex determined by $p_1, \dots, p_n$. It is clearly 0 when the vertices lie in an $(n-2)$-dimensional hyperplane. It is also clear, using the multilinearity of the determinant, that this definition of volume agrees with the definition in Section 2. (In other words, the expression for volume in Section 2 is invariant under translation of the vertices.)

Let $p_1, p_2, \dots, p_n \in L^N$ be n points and let $d_{ij}^2 = (p_i - p_j)^2$, $i \neq j = 1, 2, \dots, n$ be the squared pair-wise distances. Then we define the Cayley-Menger determinant to be

$$CM[p_1, \dots, p_n] = \det \begin{bmatrix} 0 & 1 & 1 & 1 & \cdot & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & \cdot & d_{1n}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & \cdot & d_{2n}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & \cdot & d_{3n}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & d_{1n}^2 & d_{2n}^2 & \cdot & \cdot & 0 \end{bmatrix}.$$

The algebraic condition on distances is given by the following.

Lemma 2. (Cayley-Menger) *Suppose, for a positive integer N, $p_1, p_2, \dots, p_n \in L^N$ are n points, where L is any field of characteristic not $2, 3, \dots, n-1$. Then*

$$CM[p_1, \dots, p_n] = (-1)^n 2^{n-1} ((n-1)!)^2 \operatorname{vol}^2[p_1, \dots, p_n],$$

where vol *represents the oriented $(n-1)$-dimensional volume.*

This result can be found in Blumenthal [1], page 98.

Corollary 3. *If $p_1, p_2, p_3, p_4, p_5 \in L^3$ for any field L not of characteristic 2 or 3, then*

$$CM[p_1, p_2, p_3, p_4, p_5] = 0.$$

Corollary 4. *If $p_1, p_2, p_3, p_4 \in L^3$ for any field L not of characteristic 2 or 3, then*

$$CM[p_1, p_2, p_3, p_4] = 2^3 6^2 \operatorname{vol}^2[p_1, p_2, p_3, p_4] = 2(12 \operatorname{vol}[p_1, p_2, p_3, p_4])^2,$$

where vol *is the 3-dimensional volume, and*

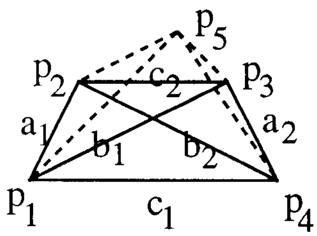
$$(12 \operatorname{vol}[p_1, p_2, p_3, p_4])^2 \in \mathbb{Z}[\dots, d_{ij}^2, \dots].$$

Proof. The coefficient of each term of $CM[p_1, p_2, p_3, p_4]$ is divisible by 2. (Each term in the expansion of the determinant is repeated when all the matrix entries are reflected about the main diagonal. These are distinct terms since if the entries in a term are reflected into each other, one entry must be fixed along the main diagonal and be 0 since the matrix has an odd number of rows and columns.)

5. The Key Lemmas

We need to control the behavior of a place when it is defined on various extensions of our base ring R. Recall that in K, $(p_i - p_j)^2 \neq 0$ for all $i \neq j$, because of the independence of their coordinates.

Lemma 3. *Let p_1, p_2, p_3, p_4, p_5 be a configuration of 5 points in K^3. Define*



$$a_1 = d_{12}^2 \qquad a_2 = d_{34}^2$$
$$b_1 = d_{13}^2 \qquad b_2 = d_{24}^2$$
$$c_1 = d_{14}^2 \qquad c_2 = d_{23}^2$$

Let φ be a place defined on the field generated by all d_{ij}^2, $i \neq j$, $i, j = 1, 2, 3, 4, 5$ such that

$$\varphi(b_1) = \varphi(b_2) = \varphi\left(\frac{b_1}{a_1}\right) = \varphi\left(\frac{b_2}{a_2}\right) = \infty,$$

and $\varphi(d_{i5}^2)$, $i = 1, 2, 3, 4$ and $\varphi(c_2)$ are finite. Then $\varphi\left(\frac{c_1}{b_1 b_2}\right) \neq 0$ and hence $\varphi\left(\frac{c_1}{b_1}\right) = \varphi\left(\frac{c_1}{b_2}\right) = \infty$.

Proof. By Corollary 3 we have

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a_1 & b_1 & c_1 & d_{15}^2 \\ 1 & a_1 & 0 & c_2 & b_2 & d_{25}^2 \\ 1 & b_1 & c_2 & 0 & a_2 & d_{35}^2 \\ 1 & c_1 & b_2 & a_2 & 0 & d_{45}^2 \\ 1 & d_{15}^2 & d_{25}^2 & d_{35}^2 & d_{45}^2 & 0 \end{bmatrix} = 0$$

We divide each circled row and column by the circled entry in that row or column, b_1 or b_2. This gives the following:

$$\det \begin{bmatrix} 0 & \frac{1}{b_1} & 1 & 1 & \frac{1}{b_2} & 1 \\ \frac{1}{b_1} & 0 & \frac{a_1}{b_1} & \textcircled{1} & \frac{c_1}{b_1 b_2} & \frac{d_{15}^2}{b_1} \\ 1 & \frac{a_1}{b_1} & 0 & c_2 & \textcircled{1} & d_{25}^2 \\ 1 & \textcircled{1} & c_2 & 0 & \frac{a_2}{b_2} & d_{35}^2 \\ \frac{1}{b_2} & \frac{c_1}{b_1 b_2} & \textcircled{1} & \frac{a_2}{b_2} & 0 & \frac{d_{45}^2}{b_2} \\ 1 & \frac{d_{15}^2}{b_1} & d_{25}^2 & d_{35}^2 & \frac{d_{45}^2}{b_2} & 0 \end{bmatrix} = 0$$

If $\varphi\left(\frac{c_1}{b_1 b_2}\right) = 0$, then all the matrix entries of $\frac{1}{b_1^2 b_2^2} CM[p_1, p_2, p_3, p_4, p_5]$ would have a finite φ value and each entry in the circled row or column would be 0 except the circled entry 1. Expanding the determinant in each of these rows or columns gives the determinant

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

so applying φ to both sides we get a contradiction to Corollary 3. Thus we get that $\varphi\left(\frac{c_1}{b_1 b_2}\right) \neq 0$.

We now prepare for the analysis of the behavior of a place φ in a neighborhood of a point in a surface S.

Lemma 4. *Suppose that $p_1, \ldots, p_n, p_{n+1}$ are $n+1$ distinct points in K^3 and that φ is a place defined on the field generated by all the pair-wise non-zero distances d_{ij}^2, $i \neq j$, $i, j = 1, \ldots, n, n+1$, such that*

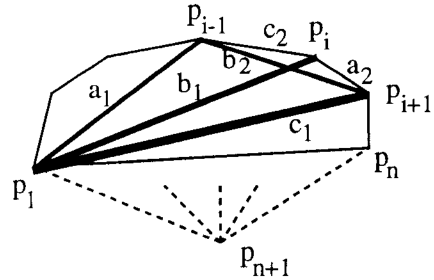
$$\varphi(d_{i,n+1}^2) \textit{ and } \varphi(d_{i,i+1}^2) \textit{ are finite for } i = 1, \ldots, n \ (i+1 \bmod n).$$

Then for some i, $\varphi(d_{i,i+2}^2)$ is finite, $i = 1, 2, \ldots, n-2$.

Proof. Suppose $\varphi(d_{i,i+2}^2) = \infty$ for $i = 1, \ldots, n-2$. We will find a contradiction. We will show inductively for $i = 3, \ldots, n$,

$$\varphi(d_{1,i}^2) = \varphi\left(\frac{d_{1,i}^2}{d_{1,i-1}^2}\right) = \infty.$$

This is clear for $i = 3$. We assume it for i and then we will show it for $i+1$. Apply Lemma 3 to



$a_1 = d^2_{1,i-1}$ $\quad a_2 = d^2_{i,i+1}$

$b_1 = d^2_{1,i}$ $\quad b_2 = d^2_{i-1,i+1}$

$c_1 = d^2_{1,i+1}$ $\quad c_2 = d^2_{i-1,i}$

The conclusion of Lemma 3 is that

$$\varphi\left(\frac{c_1}{b_1}\right) = \varphi\left(\frac{d^2_{1,i+1}}{d^2_{1,i}}\right) = \varphi(d^2_{1,i+1}) = \infty.$$

This is the inductive step. But this is the desired contradiction since for $i = n$, we get $\varphi(d^2_{1,n}) = \infty$, which contradicts $\varphi(d^2_{1,n})$ finite. Thus finally some $\varphi(d^2_{i,i+2})$ is finite.

Remark. Notice that for $n > 4$, $\varphi(d^2_{n-1,1})$ and $\varphi(d^2_{2,n})$ are not mentioned as being forced to be finite. This extra bit of generality comes for free in the proof. However, in the application of Lemma 4, it is enough to know only that some $\varphi(d^2_{i,i+2})$ is finite, where all indices are reduced modulo n. For $n = 4$, there is no extra generality.

6. The complexity of a two-manifold

In order to describe the induction steps to follow, we define a partial ordering, which we call *complexity*, for the combinatorial types of closed, orientable, triangulated two-manifolds M. Let M, N be two such two-manifolds. If the genus of M is less than the genus of N, then we say M has less complexity than N. If the genus of M equals the genus of N, and M has fewer vertices than N, then we say M has less complexity than N. If the genus of M is equal to the genus of N, M and N have the same number of vertices and the minimal degree of a vertex of M is less than the minimal degree of a vertex of N, then we say M has less complexity than N. The proof of the main theorem will be based on the complexity of M, the underlying two-manifold. It is clear that any given two-manifold can have only a finite number of other two-manifolds of comparable strictly less complexity in any given chain. It is also clear that the tetrahedron with 4 vertices and 4 triangles has strictly less complexity than any other such two-manifold. Corollary 4 allows us to start the induction.

7. Surgery

Suppose that there are three vertices i, j, k of an orientable, closed, triangulated two-manifold such that all three edges $[i, j]$, $[j, k]$ and $[k, i]$ are edges of M. If the triangle $[i, j, k]$ is *not* part of the triangulation of M, then we say that the three edges form a *splitting triangle* T for M.

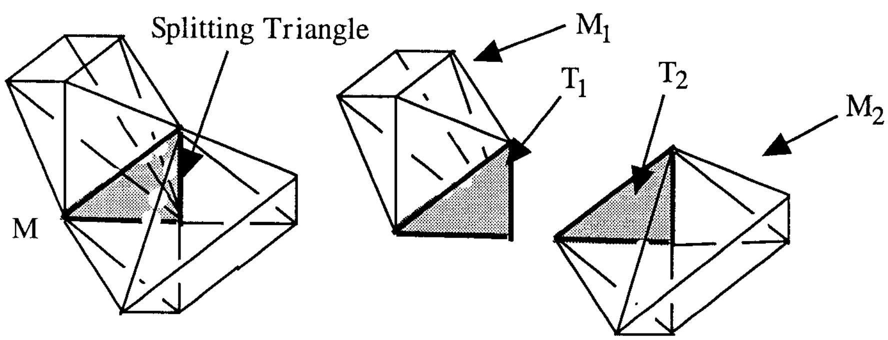
Topologically T forms a simple closed curve in M which separates a neighborhood into two components. We now describe how to do *surgery* along T. Remove the vertices i, j, k and replace them with two triangles T_1, T_2 that are each in a new triangulation of a new manifold M'. T_1 is joined with the vertices of M in one component of the neighborhood of T and T_2 is joined with the other. We call M' the result of doing surgery on M along T.

Lemma 5. *The complexity of each component of M' the surgered manifold is strictly less than the complexity of M.*

Proof. There are essentially two cases.

Case I: T separates M into two components M_1, M_2. Here the genus of $M_i \cup T_i$ is no larger than the genus of M for $i = 1, 2$. But both M_1 and M_2 have fewer vertices than M and thus less complexity.

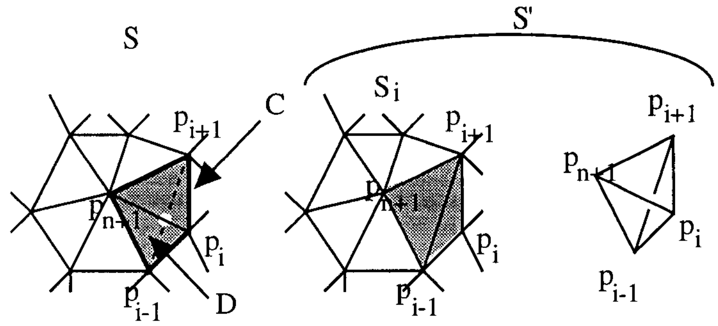
Case II: T does not separate M. Here M' is connected but the genus of M' is one less, and thus the complexity is strictly less.



We also need to consider what happens to the volume function when we do surgery along a simple closed curve C in a manifold M. By *surgery* we mean we cut M along C and add a triangulated disk D (with the consistant orientation on each triangle in D) to one local component and $-D$ (that is D with the opposite orientation) to the other local component, obtaining the surgered manifold M'. Let S be the (possibly singular) surface corresponding to M, and S' be the (possibly singular) surface corresponding to M'. Note that M' may consist of two components in which case the volume function is the sum of the volume of each component. In any case we have the following.

Lemma 6. $\mathrm{vol}(S) = \mathrm{vol}(S')$.

Proof. Each triangle in the disk D contributes one extra term in the calculation of the (generalized) volume, and the negative of that term appears in $-D$. Thus the volume remains the same.



8. The proof of the Main Theorem

We will proceed by induction on the complexity of the underlying manifold M. We will assume the statement of the theorem for all manifolds of strictly smaller complexity than M, and for all fields L. However, for the induction step we will assume that the underlying field for M and the corresponding singular surface S is the field K described in Section 2, where all the p_i, $i = 1, \ldots, N$ are distinct. After we have shown that $12\,\mathrm{vol}(S)$ is integral over R for K, we can recover the statement of the theorem for a general field by the Remark.

The induction starts with the tetrahedron, which has the smallest complexity. The corresponding surface S for the tetrahedron has $12\,\mathrm{vol}(S)$ integral by Corollary 4. We will assume that every manifold with comparable smaller complexity has the property that its corresponding surface S has an integral $12\,\mathrm{vol}(S)$. Since there are only a finite number of manifolds that we can obtain with splitting along a triangle or exchanging edges (described later), we will be done when we show $12\,\mathrm{vol}(S)$ is integral over R with our assumption.

If M has any splitting triangle T, then the surgered manifold has each component M_1, M_2 (possibly only one component) of smaller complexity. Thus each corresponding surface S_1, S_2 has $12\,\mathrm{vol}(S_i)$, $i = 1, 2$ integral over R. Since $\mathrm{vol}(S) = \mathrm{vol}(S_1) + \mathrm{vol}(S_2)$ by Corollary 2, $12\,\mathrm{vol}(S)$ is integral over R.

Suppose M has no splitting triangle. Let p_{n+1} be a point corresponding to a vertex of M with minimum degree n. Suppose $p_1, \ldots, p_n$ are the other points in order corresponding to the adjacent vertices. By our assumption about no splitting triangles, $[i-1, i+1]$ is not in the triangulation M, $(i-1, i+1 \bmod n)$. Thus we can replace any edge $[i-1, i+1]$ with $[i, n+1]$ and the triangles $[i-1, i, n+1]$, $[i, i+1, n+1]$ with $[i-1, i+1, n+1]$, $[i-1, i, i+1]$ to get another orientable two-manifold M_i and a corresponding surface S_i. Each M_i has a smaller complexity than M, so $12\,\mathrm{vol}(S_i)$ is integral over $R[d^2_{i-1,i+1}]$. See the Figure at the end of Section 7.

Let φ be any place defined on the field K that is finite on R. Consider its restriction to the field generated by $R[d^2_{1,3}, \ldots, d^2_{i-1,i+1}, \ldots, d^2_{n-2,n}]$ and denote the result by the same letter φ. By Lemma 4, $\varphi(d^2_{i-1,i+1})$ is finite for some i. Thus φ is finite over $R[d^2_{i-1,i+1}]$. By the induction hypothesis, $\varphi(12\,\mathrm{vol}(S_i))$ integral over $R[d^2_{i-1,i+1}]$. Thus $\varphi(12\,\mathrm{vol}(S_i))$ is finite. By Corollary 4, $\varphi(12\,\mathrm{vol}([p_{i-1}, p_i, p_{i+1}, p_{n+1}])$ is finite as well. Since $\mathrm{vol}(S) =$

$\mathrm{vol}(S_i) \pm \mathrm{vol}([p_{i-1}, p_i, p_{i+1}, p_{n+1}])$ we see that $\varphi(12\,\mathrm{vol}(S))$ is also finite. Thus, by Lemma 1, $12\,\mathrm{vol}(S)$ is integral over R and this finishes the proof of the Theorem.

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