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# A Purely Geometric Proof of the Uniqueness of a Triangle With Prescribed Angle Bisectors

Victor Oxman

**Abstract**. We give a purely geometric proof of triangle congruence on three angle bisectors without using trigonometry, analysis and the formulas for triangle angle bisector length.

It is known that three given positive numbers determine a unique triangle with the angle bisectors lengths equal to these numbers [1]. Therefore two triangles are congruent on three angle bisectors. In this note we give a pure geometric proof of this fact. We emphasize that the proof does not use trigonometry, analysis and the formulas for triangle angle bisector length, but only synthetic reasoning.

**Lemma 1.** *Suppose triangles $ABC$ and $AB'C'$ have a common angle at $A$, and that the incircle of $AB'C'$ is not greater than the incircle of $ABC$. If $C' > C$, then the bisector of $C'$ is less than the bisector of $C$.*

*Proof.* Let $CF$ and $C'F'$ be the bisectors of angles $C$, $C'$ of triangles $ABC$, $AB'C'$. Assuming $C' > C$, we shall prove that $C'F' < CF$.

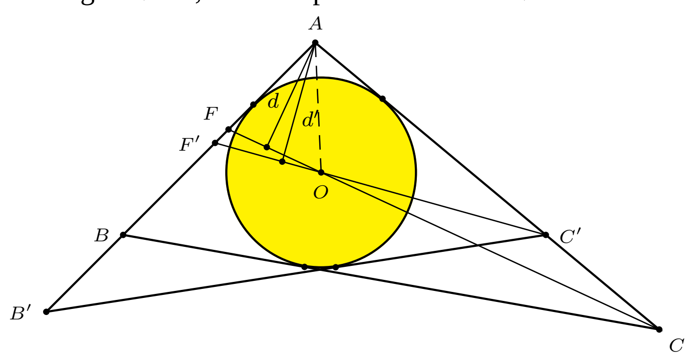


Figure 1.

Case 1. The triangles have equal incircles (see Figure 1). Without loss of generality assume $B > B'$ and the point $C'$ between $A$ and $C$. Let $O$ be the center of the common incircle of the triangles. It is known that $OF < OC$ and $OF' < OC'$. Hence, in areas,

$$\triangle OFF' < \triangle OCC'. \tag{1}$$

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Let $d$, $d'$ be the distances of $A$ from the bisectors $CF$, $C'F'$ respectively. Since $\angle AOF' = \angle OAC' + \angle AC'O = \frac{A+C'}{2} < 90°$, we have $\angle AOF < \angle AOF' < 90°$, and $d < d'$. Now, from (1), we have

$$\triangle OFF' + \triangle OC'AF < \triangle OCC' + OC'AF.$$

This gives $\triangle AF'C' < \triangle AFC$, or $\frac{1}{2}d' \cdot C'F' < \frac{1}{2}d \cdot CF$. Since $d < d'$, we have $C'F' < CF$.

Case 2. The incircle of $AB'C'$ is smaller than the incircle of $ABC$ (see Figure 2). Since the incircle of $AB'C'$ is inside triangle $ABC$, we construct a tangent $B''C''$ parallel to $BC$ that is closer to $A$ than $BC$. Let $C''F''$ be the bisector of triangle $AB''C''$. We have $C''F''||CF$ and

$$C''F'' < CF. \tag{2}$$

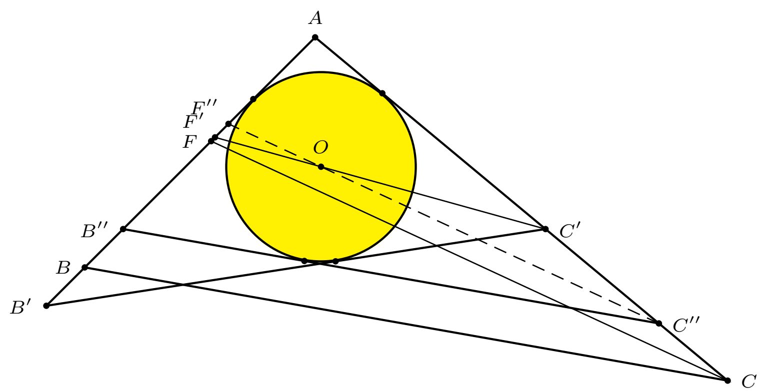


Figure 2.

Since $\angle AC''B'' = \angle ACB < \angle AC'B'$, from Case 1 we have

$$C'F' < C''F'' \tag{3}$$

From (2) and (3) we have $C'F' < CF$. □

**Lemma 2.** *Suppose triangles $ABC$ and $AB'C'$ have a common angle at $A$, and a common angle bisector $AD$, the common angle not greater than any other angle of $AB'C'$. If $C' > C$, then the bisector of $C'$ is less than the bisector of $C$.*

*Proof.* If the incirle of triangle $AB'C'$ is not greater than that of $ABC$, then the result follows from Lemma 1.

Assume the incircle of $AB'C'$ greater than the incircle of $ABC$ (see Figure 3). The line $BC$ cuts the incircle of $AB'C'$ incircle. Hence, the tangent from $C$ to this incircle meets $AB'$ at a point $B''$ between $B$ and $B'$. Let $CF$, $C'F'$ be the bisectors of angles $C$, $C'$ in triangles $ABC$ and $AB'C'$ respectively. We shall prove that $C'F' < CF$.

Consider also the bisector $CF''$ in triangle $AB''C$. Since $B$ is between $A$ and $B''$, $F$ is between $A$ and $F''$. From lemma 1 we have

$$C'F' < CF'' \tag{4}$$

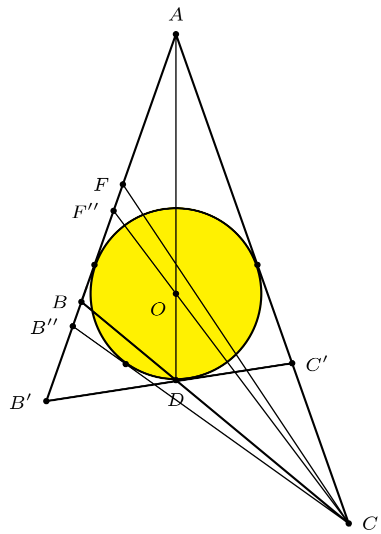


Figure 3.

Since $\angle CB''A > \angle C'B'A \geq \angle B'AC'$, we have $\angle CF''A > 90°$, and from triangle $CFF''$

$$CF'' < CF. \tag{5}$$

From (4) and (5) we conclude that $C'F' < CF$. $\square$

Now we prove the main theorem of this note.

**Theorem 3.** *If three internal angle bisectors of triangle $ABC$ are respectively equal to three internal angle bisectors of triangle $A'B'C'$, then the triangles are congruent.*

*Proof.* Denote the angle bisectors of $ABC$ by $AD$, $BE$, $CF$ and let $AD = A'D'$, $BE = B'E'$, $CF = C'F'$.

If for the angles of the triangles we have $A = A'$, $B = B'$, $C = C'$, then from the similarity of $ABC$ with $A'B'C'$ and of $ABD$ with $A'B'D'$ we conclude the congruence of $ABC$ with $A'B'C'$.

Let $A'$ be an angle that is not greater than any other angle of triangles $A'B'C'$ and $ABC$. We construct a triangle $AB_1C_1$ congruent to $A'B'C'$ that has $AD$ as bisector of angle $B_1AC_1$.

If $A' = A$ and $C' > C$, then the triangles $ABC$ and $AB_1C_1$ satisfy the conditions of Lemma 2. It follows that $C'F' < CF$, a contradiction.

If $A' < A$ and the lines $AB_1$, $AC_1$ meet $BC$ at the points $B_2$, $C_2$ respectively, without loss of generality we assume $C_1$ between $A$ and $C_2$, possibly coinciding with $C_2$ (see Figure 4). Suppose the bisector of angle $AC_2B_2$ meets $AB_2$ at $F_2$ and $AB$ at $F_3$. Since triangles $AB_1C_1$ and $AB_2C_2$ satisfy the conditions of Lemma 2, we have

$$C'F' \leq C_2F_2 < C_2F_3. \tag{6}$$

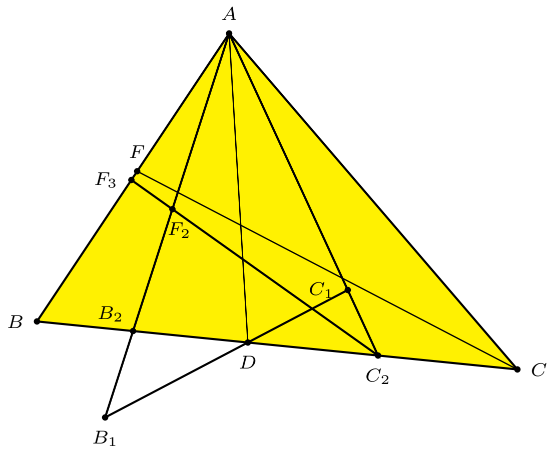


Figure 4.

The incircle of triangle $ABC_2$ is smaller than that of triangle $ABC$. Since $\angle AC_2B > \angle ACB$, by Lemma 1, $C_2F_3 < CF$ and from (6) we conclude $C'F' < CF$. This again is a contradiction. Hence, triangles $ABC$ and $A'B'C'$ are congruent. $\square$

## References


[1] P. Mironescu and L. Panaitopol, The existence of a triangle with prescribed angle bisector lengths, *Amer. Math. Monthly*, 101 (1994) 58–60.



Victor Oxman: Western Galilee College, P.O.B. 2125 Acre 24121 Israel
*E-mail address*: `victor.oxman@gmail.com`