ON FENCHEL'S THEOREM

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Fenchel's well-known theorem on the total curvature μ of a closed C^2 space curve consists of the inequality $\mu \geqq 2\pi$, together with the statement that equality holds if and only if the curve is a plane convex curve. Our purpose is to deduce this result with a symmetry argument that seems simpler than any proof appearing in the textbooks or recent literature.

Let $G = \{R(s) \mid 0 \leqq s \leqq L\}$ be a closed C^2 space curve with arclength parameter s, unit tangent $T(s) = (T_1, T_2, T_3) \equiv (d/ds)R(s)$, and curvature

$$|\kappa(s)| \equiv \|(d/ds)T(s)\|.$$

The ***tangent indicatrix*** of G is the curve $\Gamma = \{T(s) \mid 0 \leqq s \leqq L\}$ on the unit sphere S, and the ***total curvature*** of G is the quantity $\mu \equiv \int_0^L |\kappa(s)|\, ds$. Clearly, the total curvature of G is just the length of its tangent indicatrix. Consider the following two lemmata:

LEMMA 1. *The tangent indicatrix Γ of a closed C^1 space curve G is not contained in any open hemisphere of S. It is contained in a closed hemisphere if and only if G is a plane curve.*

Proof: If Γ were contained in a hemisphere of S, we could perform a rotation to place it in the northern hemisphere. Thus, we may assume $T_3(s) \geqq 0$ for all $s \in [0, L]$ and since G is a closed curve we know that

$$0 = (R(L) - R(0))_3 = \left(\int_0^L T(s)ds\right)_3 = \int_0^L T_3(s)ds = 0.$$

This shows that $T_3(s)$ cannot be strictly positive, and hence Γ cannot lie in an open hemisphere. Furthermore, since $T_3(s)$ is nonnegative it must vanish identically, i.e., $0 \equiv T_3(s) \equiv (d/ds)R_3(s)$, and hence G must lie in a plane $R_3 = \text{constant}$. Conversely, if G is a plane curve, then Γ lies on a great circle and hence is contained in a closed hemisphere.

LEMMA 2. *Let Γ be a closed rectifiable curve on S. If the length of Γ is less than 2π, then it is contained in an open hemisphere of S; if Γ has length 2π, then it is contained in a closed hemisphere of S.*

Proof. Let P be any point on Γ and let Q be the point of Γ such that the curve segments $\Gamma_1 = PQ$ and $\Gamma_2 = QP$ have equal length, $\Gamma = \Gamma_1 + \Gamma_2$. Rotate S so that P and Q are located symmetrically with respect to the north pole N, i.e., so that either $P = Q = N$ or so that P and Q have the same latitude but have longitudes differing by 180°. If Γ does not now intersect the equator, the conclusions follow. If Γ_1 intersects the equator at some point, construct the unique curve Γ_2' which is symmetric to Γ_1 with respect to N. Then Γ_2' has the same length as Γ_1, the closed curve $\Gamma' \equiv \Gamma_1 + \Gamma_2'$ has the same length as Γ, and there is a pair of antipodal equatorial points on Γ'. But if we join these points with great semicircles (which are the geodesics on S), we see that if Γ_1 intersects the equator, then Γ' (and hence Γ) has length at least 2π, and that if Γ_1 crosses the equator into the open southern hemisphere, then Γ' must be strictly longer than 2π.

Thus, if Γ has length less than 2π, then Γ_1 cannot intersect the equator and if the length of Γ is exactly 2π, then Γ_1 cannot cross the equator. Since the same argument applies to Γ_2 we conclude that if Γ has length less than 2π, then it is contained in the open northern hemisphere and that if its length is exactly 2π, then it must lie in the closed northern hemisphere.

Fenchel's inequality now follows immediately from these two results: Since the tangent indicatrix of a closed C^2 space curve cannot lie in an open hemisphere of S its length must be at least 2π. If the length is exactly 2π, then the tangent indicatrix lies in a closed hemisphere of S and hence the original curve must be a plane curve. Using the notion of the *rotation index* as in [1], one can now complete the full proof of Fenchel's Theorem by showing that a plane curve is convex if and only if its tangent indicatrix has length 2π.

REMARKS: This proof grew out of a discussion with Laird E. Taylor about a problem posed by Robert Osserman. It is a pleasure to acknowledge their helpful suggestions as well as those of S. S. Chern, H. Flanders, and the referees.

Contrary to our original belief, this treatment is not the first completely elementary proof of Fenchel's inequality. The referees have remarked that Anthony Morse, A. S. Besicovitch, and H. Flanders have presented elementary proofs in their lectures, and that shortly after Fenchel's original publication [2] in 1929, H. Liebmann [6] published a similar elementary proof which has been almost totally forgotten. In a 1951 article, Fenchel [3] gives references to several different, less elementary, proofs, while recent textbooks such as [4] and [5] have followed a surface theory approach, introduced by K. Voss [7] in 1955. It is interesting to compare these proofs with ours and with each other; Fenchel's elegant pearl has certainly inspired a variety of settings.

This work was partially supported by the National Science Foundation under grant GP-13258.

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