TETRAHEDRAL SHOELACE METHOD: A METHOD TO CALCULATE VOLUME OF IRREGULAR SOLIDS

Nicholas Patrick

Supervisor: Nadya Pramita

SMP Cita Hati Christian School, Surabaya - Indonesia, 2nicholaspatrick2@gmail.com

1. Introduction

Calculating the volume of regular and irregular solids is an important task in nearly all branches of science and also in real life applications. Since the story about Archimedes, we have been calculating volumes with different methods such as water displacement, convex polyhedron calculation, density and mass measurement, integration, and solving by parts. All those methods have some limitations: water displacement and density and mass measurement needs waterproof and physical and relatively small objects, solving by parts only works with simple shapes, convex polyhedra volume calculation only works for convex hulls, and integration only works with solids of revolution.

As a student of Mathematics, I have been challenged to try and find another effective way to calculate the volume of any solid with one formula.

2. Research Method

This method takes inspiration from The Shoelace Formula; a method of calculating area given the Cartesian coordinates of the vertices, and applies it in 3D. The first task is to divide a given shape into 3D counterparts of triangles, tetrahedra, where taking a triangular surface and a common origin point forms each tetrahedron. The next step is to decide whether the tetrahedra should be added to ($Ptet$) or subtracted from ($Ntet$) the final result. We can intuitively tell that a ray emanating from the origin in The Shoelace Formula can tell what lengths will be calculated as area; as the ray goes through a side, it will start/stop counting area. The same goes for 3 dimensions. So if we set the sign of the volume to be the sign of the derivative of the points (facing in/out, can be told by the right hand rule), we can instantly differentiate $Ptet$ and $Ntet$. With this, we can see that a Shoelace Formula in 3D will need to have the same direction for all the surfaces. A polyhedron can be triangulated into tetrahedra whose volume can be calculated using The Tetrahedral Shoelace Method. If all the directions of the surfaces in the determinants are the same then two surfaces joining together ($Ptet - Ntet$) will cancel out, leaving only the surface of the polyhedron's tetrahedra to be calculated. The volume of a single tetrahedron is given by the formula

$$V = \frac{1}{6} \times \left| det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \right|$$

Fig.1. Volume of a single tetrahedron.

So the volume of any polyhedron can be given by the Tetrahedral Shoelace Formula (Fig. 2.)

$$V = \frac{1}{6} \left| \sum_{Surfaces} \det \begin{bmatrix} CoP_1 \\ CoP_2 \\ CoP_3 \end{bmatrix} \right|$$

Fig.2. Tetrahedral Shoelace Formula CoP is Coordinates of Point

After all the math is checked, a computer program using those formulas can calculate the volume of any given polyhedron.

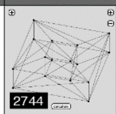
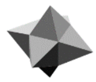
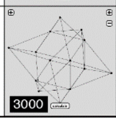
Solids	Regular vol. calculation (as a reference)	Tetrahedral Shoelace Method calculation program in JS*	
 Toroidal Polyhedron	8 cubes in an O shape length of each cube = 7 Vol. of each cube = 7^3=343 Total Vol. = 343*8= **2744**	 2744	match
 Tetrakis Hexahedron	length of cube=10 Cube vol -> 10^3 =1000 Height of pyramids=10 1/3*10^2*10=333.333 6 pyramids -> 6*333.333= 2000 Vol.= 1000+2000 = **3000**	 3000	match

Fig.3. Table of comparison *JavaScript program by Nicholas Patrick

3. Result and Analysis

The Tetrahedral Shoelace Method can calculate the volume of any irregular solid by making a polyhedral approximation and calculate that shape's volume. For higher accuracy, more vertex coordinates are required. So, for simple shapes, small physical waterproof shapes, convex hulls, or shapes of revolution, we may still use recent methods. But for other objects where it is possible to find the Cartesian coordinates of the vertices, one may use the Tetrahedral Shoelace Method.

4. Conclusion

It can be concluded that this method can calculate volumes of solids of physical/abstract, convex/non-convex, simple/complex, revolutional/non-revolutional and shapes with/without hole(s). This method can calculate the volume of any solid with one formula and can be applied as a complement of recent methods.

5. References

[1] Varberg, D., Purcell, E., & Rigdon, S. (2006). Calculus (9th ed.). (pp. 574–578). London, UK: Pearson.

[2] Bart Braden (1986). The Surveyor's Area Formula (pp. 326–329) The College Mathematics Journal.

[3] Schmaltz, Wolfgang. (2009). A Jordan–Brouwer Separation Theorem (PDF). (pp: 1–19). Chicago: University Paper.