Two Important Lemmas in Olympiad Geometry

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Introduction

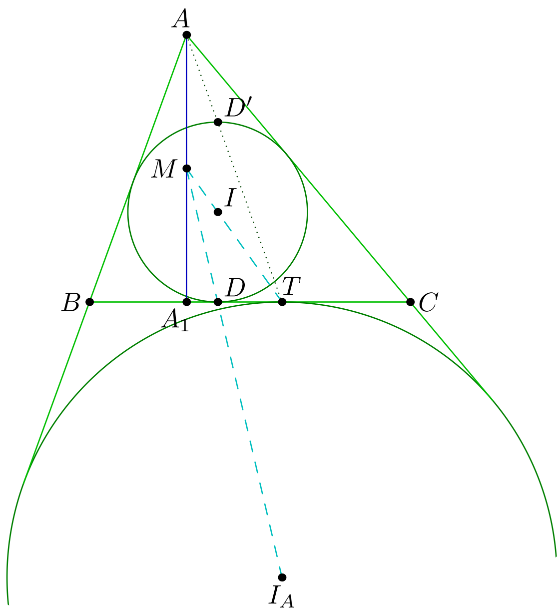
In this article, I present the key lemmas for the legendary Iran TST 2009 problem 9, which is famous enough to have an entire configuration named for it. I include the midpoint of altitudes lemma and the right angle on incircle chord lemma, which are both crucial to the Iran problem. There are also five practice problems at the end.

§1 Midpoint of Altitudes

The midpoint of altitudes configuration involves two key collinearities, described in the following lemma.

Lemma 1.1 (Midpoint of Altitudes)
In triangle ABC with incenter I and A-excenter I_A, let D and T be the incircle and A-excircle tangency point with $\overline{BC}$. If M is the midpoint of the A-altitude, then M, I, T collinear and M, D, I_A collinear.

Proof. The proof is not difficult, but it is slightly tricky. Here I will only prove M, I, T collinear since the other one is analogous.



This first thing to note is that A, D', T collinear, where D' is the antipode of D in the incircle. Indeed, the homothety at A sending the incircle to the excircles maps D' to T since they are both "top points."

Now consider the homothety at T sending $\overline{A_1A}$ to $\overline{DD'}$. Where does M go? Since it's the midpoint of $\overline{AA_1}$, it is sent to the midpoint of $\overline{DD'}$, which is precisely I! $\square$

§2 Incenter Perpendicularity Lemma

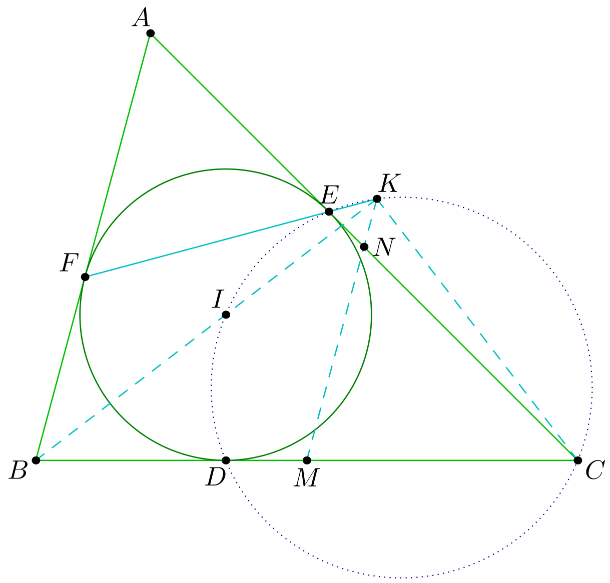
This incenter perpendicularity is slightly trickier than the previous result.

Lemma 2.1 (Right Angle on Incircle Chord)
The incircle of $\triangle ABC$ is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F respectively. Let M and N be the midpoints of $\overline{BC}$ and $\overline{AC}$. If K is the intersection of lines BI and EF, then $\overline{BK} \perp \overline{KC}$. In addition, K lies on line MN.

Proof. The first part can be reduced to showing that pentagon $CDIEK$ is cyclic. Luckily, this is pretty simple. Since D and F are reflections across line BI, we have

$$\angle KDC = 180^\circ - \angle BDK = 180^\circ - \angle BFK = \angle AFE = \angle AEF = \angle KEC,$$

which completes the proof.



For the second part, observe that M is the circumcenter of $\triangle BKC$. Thus $\angle CMK = 2\angle KBC = \angle B$, so $\overline{MK} \parallel \overline{AB}$. Since $\overline{MN} \parallel \overline{AB}$, it follows that K lies on line MN. $\square$

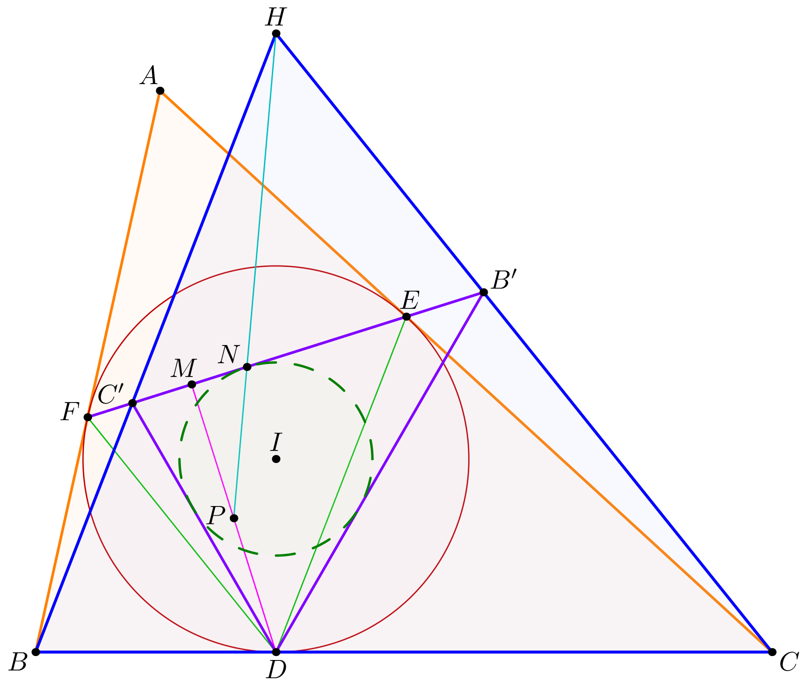
§3 The Legendary Iran TST

Now we are finally ready to tackle the Iran TST problem. While the problem is quite difficult without knowing such lemmas, the solution with them is actually quite short!

Example 3.1 (Iran TST 2009/9)

In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. Let M be the foot of the perpendicular from D to $\overline{EF}$, and let P be on $\overline{DM}$ such that $DP = MP$. If H is the orthocenter of $\triangle BIC$, prove that $\overline{PH}$ bisects $\overline{EF}$.

Proof. Let B', C' be the feet of the altitudes from B and C in triangle HBC, and let N be the midpoint of $\overline{EF}$.



By lemma 2.1, we know that B' and C' lie on $\overline{EF}$. Also, since H is the orthocenter of $\triangle BIC$, it is also the D-excenter in $\triangle DB'C'$.

The key is then that I is the orthocenter of $\triangle HBC$, so it is the incenter of $\triangle DB'C'$. Thus N is the a tangency point of the incircle with $B'C'$. Then a direct application of lemma 1.1 finishes the problem. $\square$

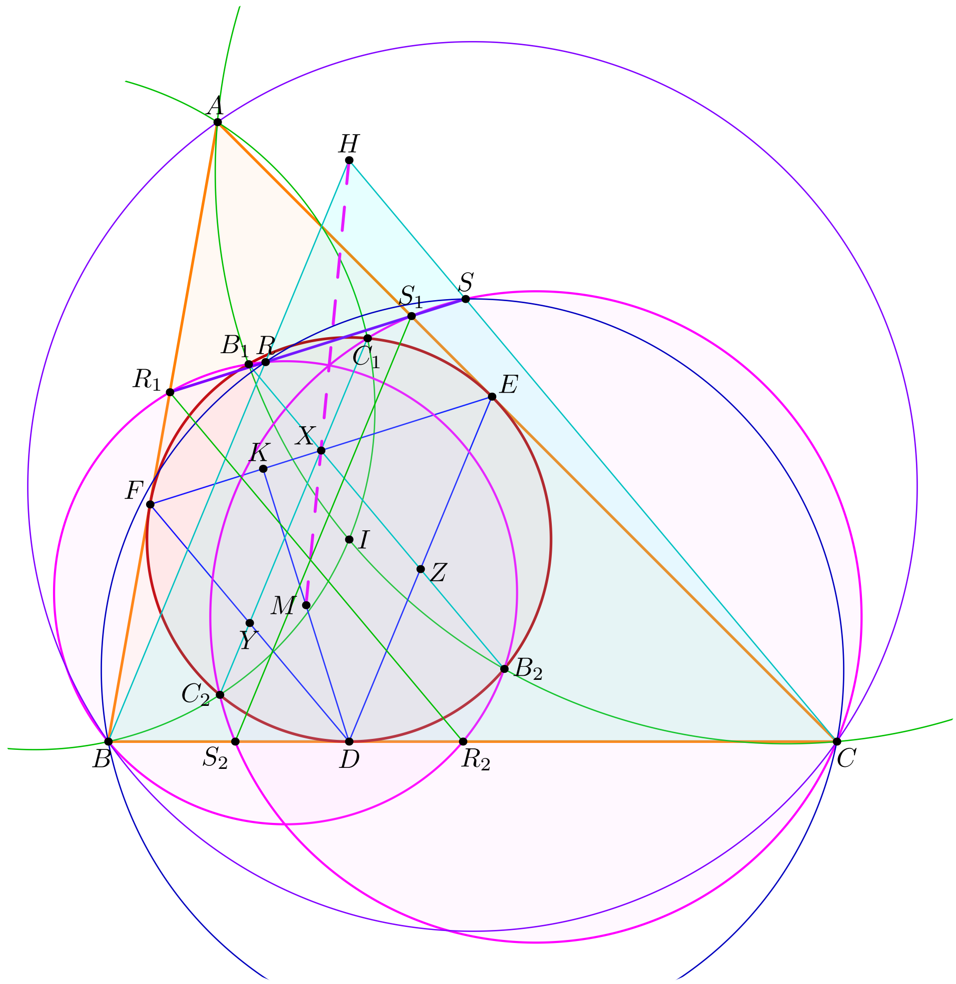
§4 Grand Finalé: TSTST 6

To conclude this article, I will present the inspiration for this handout and one of the most difficult geometry problems that I have seen. This beautiful solution was found by AoPS users **anantmudgal09** and **EulerMacaroni**.

Example 4.1 (USA TSTST 2016/6)

Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

Proof. First let's do some labeling. Let X, Y, Z be the midpoints of $\overline{EF}$, $\overline{FD}$, $\overline{DE}$, and let the circumcircles of triangles BB_1B_2 and CC_1C_2 be ω_B and ω_C. Also, if H is the orthocenter of $\triangle BIC$, let ω_B intersect $\overline{HB}$, $\overline{AB}$, $\overline{BC}$ at R, R_1, R_2. Similarly, let ω_C intersect $\overline{HC}$, $\overline{AC}$, $\overline{BC}$ at S, S_1, S_2.



Observe that X is the radical center of (AEF), (DEF), (AIB), (AIC), so X is in fact the intersection of $\overline{B_1B_2}$ and $\overline{C_1C_2}$. We can obtain similar results for Y and Z; thus, $\overline{XZ}$ and $\overline{XY}$ are just $\overline{B_1B_2}$ and $\overline{C_1C_2}$.

We must have B_1, B_2 symmetric about line BI, so ω_B is also symmetric about line BI. So R_1 and R_2 are reflections across $\overline{BI}$, so $\overline{R_1R_2} \parallel \overline{DF}$. Similarly, $\overline{S_1S_2} \parallel \overline{DE}$.

The key is the following claim.

Claim. R, R_1, S, S_1 are collinear.

Proof. To prove this, we do some angle chasing. We know that $\angle R_2R_1R = \angle R_2BR = \angle DFE$ and $\angle S_2S_1S = \angle 180^\circ - \angle S_2CS = 180^\circ - \angle DEF$. Finally, we have $\angle(R_1R_2, S_1S_2) = \angle EDF$ from the parallel lines, so

$$\angle R_2R_1R + 180^\circ - \angle S_2S_1S + \angle(R_1R_2, S_1S_2) = 180^\circ.$$

Thus $\overline{R_1R}$ and $\overline{S_1S}$ must be the same line, so all of them are collinear. $\square$

Notice that $\overline{R_1RS_1S} \parallel \overline{EF}$ and these lines are both antiparallel to $\overline{BC}$, so B, C, R, S are concyclic. Therefore by Radical center on $(BCSR)$, ω_B, ω_C, we get that H lies on the Radical axis of ω_B and ω_C. However, by example 3.1, H, X, M collinear, so M also lies on the radical axis of ω_B and ω_C. This completes the proof. $\square$

§5 Problems

Problem 5.1 (Vietnam TST 2003/2)**.** In triangle ABC, let O be the circumcenter and I the incenter. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of the altitudes AH, BK, CL, respectively. The incircle of triangle ABC touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F, and OI are concurrent.

Problem 5.2 (Romania TST 2007/2)**.** Let ABC be a triangle, let E, F be the tangency points of the incircle $\Gamma(I)$ to the sides AC, respectively AB, and let M be the midpoint of the side BC. Let $N = AM \cap EF$, let $\gamma(M)$ be the circle of diameter BC, and let X, Y be the other (than B, C) intersection points of BI, respectively CI, with γ. Prove that

$$\frac{NX}{NY} = \frac{AC}{AB}.$$

Problem 5.3 (USA TST 2015/1)**.** Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

Problem 5.4 (Shortlist 2002/G7)**.** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 5.5 (Shortlist 2004/G7)**.** For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

Problem 5.6 (Taiwan TST 2015)**.** In a scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of triangle AEF at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.

§6 Hints

Do not look at the hints until you are very stuck. Some contain very big spoilers.

5.1 Direct application of lemma 1.1. Homothety to finish.

5.2 You will need the lemma that $\overline{NI} \perp \overline{BC}$. Then apply lemma 2.1 and ratio lemma to conclude.

5.3 Apply lemma 2.1 to show that P lies on $\overline{DE}$. Then angle chase.

5.4 Let the tangent to the incircle at N intersect $\overline{BC}$ at T. Then it suffices to show that $TN^2 = TB \cdot TC$. Apply lemma 1.1.

5.5 Apply lemma 2.1 to $\triangle ABX$ and $\triangle ACX$. Should be fairly straightforward from here.

5.6 Requires decent knowledge of projective geometry. First show that it suffices to show that S lies on the polar of T with respect to the nine-point circle of $\triangle BIC$. Use lemma 2.1 to help you finish.

References

[1] **Euclidean Geometry in Mathematical Olympiads** by Evan Chen for some relevant problems.

[2] **Lemmas in Olympiad Geometry** by Titu Andreescu, Sam Korsky, and Cosmin Pohoata for some other relevant problems.

[3] AoPS users **anantmudgal09** and **EulerMacaroni** for their proof of TSTST 6 at `http://artofproblemsolving.com/community/c6h1264730p8497943`.