Classifications of Frieze Groups and an Introduction to Crystallographic Groups

Tyler Landau

May 10, 2019

Abstract

We will be looking at two special infinite plane symmetry groups namely frieze and crystallographic (wallpaper) groups. Within each of these groups we aim to describe what patterns we can form, in particular what qualifications determine which of the 7 frieze or 17 wallpaper groups a given pattern is apart of. For the frieze groups, we will also look at the construction of each pattern, their isomorphism classes, and why there are only 7 of them.

0 Introduction

Frieze patterns are an interesting set of groups as their name originated from the architectural term of a frieze or a broad decorative band. This description is not far from the frieze patterns we see in group theory, where our formal definition of a frieze group is a plane symmetry group whose translations are isomorphic to $\mathbb{Z}$. Our definition for wallpaper symmetry is very similar, but in these groups the translations are isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. We will recall a symmetry group is the set of all isometries in $\mathbb{R}^n$ that carry a set of points G to itself (this group operation is function composition), and an isometry is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ which preserves distance. As we look further into the patterns we can form, we will also be classifying each group into their isomorphism classes using a property defined as the semidirect product. For semidirect products we will make use of homomorphisms and their properties as well as automorphisms of a group G ($\text{Aut}(G)$). We will also be looking at the *normalizer* of a subgroup H in G, defined with the following notation, $N_G(H) = \{g \in G | gHg^{-1} = H$. For most of the groups we will be looking at, the property of our group $G = H \times K$ where H and K are subgroups of G will not typically be satisfied. The reason we will be unable to use the internal direct product is due to the condition $hk = kh$ for all h and k not being met, i.e. we will be constructing non-abelian groups. The main group we will be constructing is the dihedral group.

Originally used in Greek architecture [1], the frieze is the section of a structure between the support beams and the top of the structure (usually a roof). The patterns started off as simply patterns of lines repeated all the way around the building, with each set of lines spaced a particular distance away from the previous one. Later on the patterns became more intricate involving moldings or painting in each of the spaces where the lines used to be, but it would still be the same image repeated all the way around the structure. Since the original development of a frieze in Greek architecture, the term has branched out to mean any long horizontal band used for decorative purposes, this could be on pottery, interiors or exteriors of walls, and many other objects.

A frieze is no longer commonly seen in architecture, but instead it is put in places we might not usually notice. Take lanyards for example; they clearly have transitional symmetries, and depending on the figure there is a wide range of symmetries that can be involved. We will be taking a close look at all of the possible symmetries to create patterns on horizontal strips such as a lanyard and then introducing patterns on two dimensions.

1 Semidirect Product

In order for us to classify our symmetry groups we need to establish a definition of a semidirect product. Recall the internal direct product of two normal subgroups H and K where $H \cap K = \{e\}$, and $G = HK$ is written as $H \times K$. Our goal with establishing the semidirect product is to generalize the direct product definition to apply to H and K where only one need be normal in G. Suppose we have a group G such that $H \trianglelefteq G$, $K \leq G$, and $H \cap K = \{e\}$. Since H and K are subgroups of G then we know HK will also be a subgroup in G, where each element $hk \in HK$ is uniquely defined i.e. there is a bijection between HK and the the ordered pairs (h, k) [2]. We will also note their product of any two elements is defined as follows:

$$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1k_1h_2k_2 \\ &= h_1k_1h_2(k_1^{-1}k_1)k_2 \\ &= h_1(k_1h_2k_1^{-1})k_1k_2 \\ &= h_3k_3.\end{aligned}$$

Since H is normal we know $k_1h_2k_1^{-1}$ is an element of H thus allowing us to conclude the last line of our product. This example is based on the assumption that we already have a group G such that $H, K \leq G$ with $H \trianglelefteq G$ and $H \cap K = 1$. We want to extend this concept to starting with the abstract groups of H and K, using them to construct our G. In order to start this construction, we need to have multiplication in G defined in terms of multiplication in H and K. From above we see our k_3 value is obtained through multiplication in K, the problem we encounter is with the element $k_1h_2k_1^{-1}$, which we will define in terms of H and K instead of referencing G. If we look at our term $k_1h_2k_1^{-1}$, we see K is acting on H by *conjugation* which we will define as:

$$k \cdot h = khk^{-1}.$$

The action defined above gives us a homomorphism $\phi : K \mapsto \mathrm{Aut}(H)$ thus showing multiplication in HK solely depends on the multiplication in K, multiplication in H, and our function ϕ. Using the investigation above we use Theorem 10 in chapter 5 from [2].

Theorem 1.1. *Let H and K be groups and let ϕ be a homomorphism from K to $\mathrm{Aut}(H)$. Let "$\cdot$" denote the (left) action of K on H determined by ϕ. Let G*

be the set of ordered pairs (h,k) with $h \in H$ and $k \in K$ and define the following multiplication on G:

$$(h_1,k_1)(h_2,k_2) = (h_1\phi(k_1)(h_2), k_1k_2) = (h_1k_1 \cdot h_2, k_1k_2).$$

This multiplication which we have defined makes G a group and naturally allows us to conclude $|G| = |H||K|$.

We also know the sets $\{(h,1)|h \in H\}$ and $\{(1,k)|k \in K\}$ are subgroups of G given the way we have defined G, in addition we know these sets are isomorphic to H and K i.e.:

$$H \cong \{(h,1)|h \in H\} \quad \text{and} \quad K \cong \{(1,k)|k \in K\}.$$

From these isomorphisms of H and K in G we see $H \trianglelefteq G$, $H \cap K = 1$, and for all $h \in H$ and $k \in K$ we have $khk^{-1} = k \cdot h = \phi(k)(h)$.

Before we start our proof we will first note a few important properties of our homomorphism ϕ. The first couple properties we notice given the definition of our homomorphism ϕ are $\phi(k)(1) = 1$ as we are strictly mapping k to the identity, and similarly we see $\phi(1)(h) = h$ as the identity element of K maps to the identity $\mathrm{Aut}(H)$. The last property of our homomorphism is a basic rule of homomorphisms that composition of our ϕ translates to multiplication in K, i.e. $\phi(k_1)(\phi(k_2)(h)) = \phi(k_1k_2)(h)$.

Proof. In order to show G is a group under the multiplication we defined, we will verify the associative law, and show the existence of the identity and inverses. For associativity we see for all (h_1,k_1), (h_2,k_2), and (h_3,k_3) in G:

$$\begin{aligned}
\Big((h_1,k_1)(h_2,k_2)\Big)(h_3,k_3) &= (h_1\phi(k_1)(h_2), k_1k_2)(h_3,k_3) \\
&= (h_1\phi(k_1)(h_2)\phi(k_1k_2)(h_3), k_1k_2k_3) \\
&= (h_1\phi(k_1)(h_2)\phi(k_1)(\phi(k_2)(h_3)), k_1k_2k_3) \\
&= (h_1\phi(k_1)(h_2\phi(k_2)h_3), k_1k_2k_3) \\
&= (h_1,k_1)(h_2\phi(k_2)(h_3), k_2k_3) \\
&= (h_1,k_1)\Big((h_2,k_2)(h_3,k_3)\Big).
\end{aligned}$$

Since H and K are groups then we know our 1 element is in both H and K. In addition G is defined as all possible ordered pairs (h,k) where h and k are in H and K respectively, so we know (1,1) must be an element of G. Now we will verify this is the identity of G:

$$\begin{aligned}
(h,k)(1,1) &= (h\phi(k)(1), k1) \\
&= (h1, k) \\
&= (h,k).
\end{aligned}$$

For inverses we will suppose $(h,k)^{-1} = (\phi(k^{-1})(h^{-1}), k^{-1})$, and to verify this we will look at the following:

$$\begin{aligned}
(h,k)(\phi(k^{-1})(h^{-1}), k^{-1}) &= (h\phi(k)(\phi(k^{-1})(h^{-1})), kk^{-1}) \\
&= (h\phi(kk^{-1})(h^{-1}), 1) \\
&= (h\phi(1)(h^{-1}), 1) \\
&= (hh^{-1}, 1) \\
&= (1,1).
\end{aligned}$$

Now we must verify $H \trianglelefteq G$, $H \cap K = 1$, and for all $h \in H$ and $k \in K$ we have $k \cdot h = khk^{-1} = \phi(k)(h)$, this will simplify the rest of our proof by giving us the notation of $k \cdot h = \phi(k)(h)$. First we will define $\tilde{H}$ and $\tilde{K}$ as follows $\tilde{H} = \{(h,1)|h \in H\}$ and $\tilde{K} = \{(1,k)|k \in K\}$, now we see the following properties of $\tilde{H}$ and $\tilde{K}$:

$$\begin{aligned}(h_1,1)(h_2,1) &= (h_1\phi(1)(h_2),1)\\ &= (h_1h_2,1)\\ (1,k_1)(1,k_2) &= (1\phi(k_1)(1),k_1k_2)\\ &= (1,k_1k_2).\end{aligned}$$

This proves $\tilde{H}$ and $\tilde{K}$ are also isomorphic to H and K respectively, as well as verifying they are indeed subgroups of G. Since we see that $\tilde{H} \cap \tilde{K} = 1$, then since we have our isomorphism from above, it is implied that $H \cap K = 1$. Now in order to verify $khk^{-1} = k \cdot h$ we will look at the following:

$$\begin{aligned}(1,k)(h,1)(1,k^{-1}) &= (\phi(k)(h),k)(1,k^{-1})\\ &= (\phi(k)(h)\phi(k)(1),kk^{-1})\\ &= (\phi(k)(h),1)\\ &= (k \cdot h,1).\end{aligned}$$

This verifies by isomorphism that $khk^{-1} = \phi(k)(h) = k \cdot h$.

By our definition of multiplication in G we see $K \leq N_G(H)$, and by definition $H \leq N_G(H)$. Since $G = HK$ we see $N_G(H) = G$ which implies H is normal in G. ■

Definition 1.1. *The group G described in Theorem 1.1 is called the semidirect product of H and K with respect to ϕ and is denoted by $H \rtimes_\phi K$ (when there is no confusion we will simply write $H \rtimes K$).*

A simple example of the semidirect product at work is in $\mathbb{D}_4$, which is defined in the following Cayley table:

	1	ρ	ρ^2	ρ^3	F	ρF	$\rho^2 F$	$\rho^3 F$
1	1	ρ	ρ^2	ρ^3	F	ρF	$\rho^2 F$	$\rho^3 F$
ρ	ρ	ρ^2	ρ^3	1	ρF	$\rho^2 F$	$\rho^3 F$	F
ρ^2	ρ^2	ρ^3	1	ρ	$\rho^2 F$	$\rho^3 F$	F	ρF
ρ^3	ρ^3	1	ρ	ρ^2	$\rho^3 F$	F	ρF	$\rho^2 F$
F	F	$\rho^3 F$	$\rho^2 F$	ρF	1	ρ^3	ρ^2	ρ
ρF	ρF	F	$\rho^3 F$	$\rho^2 F$	ρ	1	ρ^3	ρ^2
$\rho^2 F$	$\rho^2 F$	ρF	F	$\rho^3 F$	ρ^2	ρ	1	ρ^3
$\rho^3 F$	$\rho^3 F$	$\rho^2 F$	ρF	F	ρ^3	ρ^2	ρ	1

Suppose we look at the groups $H = \{1, \rho, \rho^2, \rho^3\}$ and $K = \{1, F\}$, the first thing we want to figure out what the group $\mathrm{Aut}(H)$ looks like. We notice from the Cayley table these operations are cyclic, in particular both ρ and ρ^3 are generators of H. From here we can conclude any automorphism must take 1 and

ρ^2 to themselves as they are the only elements with order 1 and 2 respectively. Thus our only *two* options for the automorphisms of H are when we map ρ and ρ^3 to themselves ($\rho \to \rho$) and when we map each of them to inversion ($\rho \to \rho^{-1}$). Thus we can define our ϕ function between K and $\text{Aut}(H)$ as $1 \to (\rho \to \rho)$ and $F \to (\rho \to \rho^{-1})$. By this definition of ϕ we see it is a homomorphism, and thus our criteria for Theorem 1.1 is met so we can find our group $G = H \rtimes_\phi K$:

$$\begin{aligned} G &= HK \\ &= \{1(1), 1(F), \rho(1), \rho(F), \rho^2(1), \rho^2(F), \rho^3(1), \rho^3(F)\} \\ &= \{1, F, \rho, \rho F, \rho^2, \rho^2 F, \rho^3, \rho^3 F\}. \end{aligned}$$

This shows how the semidirect product can be used to generate non-abelian groups such as $\mathbb{D}_4$.

Our example above looks at the semidirect product of two groups, H and K, where our H is a cyclic group (isomorphic to $\mathbb{Z}$ if infinite or $\mathbb{Z}_n$ if order is n) and our K is isomorphic to $\mathbb{Z}_2$. We will now make a generalization from this result by first noticing our ϕ function maps the non-identity element of K to the automorphism of inversion on H. We see from our example where $H \cong \mathbb{Z}_4$ (n=4) and $K \cong \mathbb{Z}_2$ that $H \rtimes K = \mathbb{D}_4 = \mathbb{D}_n$, this will always be true unless $H \cong \mathbb{Z}$, in which case we will have $H \rtimes K = \mathbb{D}_\infty$. It is important to recognize at this point that the infinite dihedral group we will be referring to is constructed with $H \cong \mathbb{Z}$ as opposed to $H \cong \mathbb{R}$. This difference in the size of our H subgroup impacts the size of our dihedral group as one is countably infinite ($\mathbb{Z}$) and the other is uncountably infinite ($\mathbb{R}$).

2 Motions on Plane Symmetries

Before we get into classifying different groups of plane symmetries, we must describe the isometries that make up these groups. The five motions we need to define are translations, rotations, glide reflections, horizontal reflections, and vertical reflections. Compositions of these motions dictates how any pattern will look, depending on the motions there will be constructions which are equivalent to another. Each of the frieze and wallpaper groups will consist of translations composed with any other motion. What separates frieze from wallpaper groups, as mentioned before, is the fact that the group of translations is isomorphic to $\mathbb{Z}$ for frieze groups and $\mathbb{Z} \oplus \mathbb{Z}$ for wallpaper. One of the features about each of the frieze and wallpaper groups is the construction of a group with certain motions naturally fulfills other motions. An example of this is comes up when we look at a group formed with both vertical and horizontal reflections. In this group we will notice rotations are automatically fulfilled, thus we do not reference them in our groups construction. We will look deeper at this idea when we show why there are only 7 possible frieze patterns with all these motions.

2.1 Translations

A translation is a linear shift of some figure along the plane. As indicated before, the translations of our two plane symmetries are of infinite order. For frieze groups there is simply one type of translation, t, that translates a figure on some linear path, but for wallpaper groups there are two different translations,

t which translates horizontally and t' which translates vertically. For each of the frieze and wallpaper groups, we will refer to the set of all translations as T.

2.2 Rotations

When we refer to a rotation symmetry we will only include non-trivial rotations around a particular point, i.e. we will only talk about rotations, r, such that $0 < r < 360°$. For frieze groups, our only possible rotation will be a rotation of $180°$ as our translations are only on a single linear path. Wallpaper groups have a lot more diversity in the rotational symmetries, which is the key reason there are more wallpaper patterns than frieze patterns. Since there are more possibilities of rotations in wallpaper groups, our r will refer to the smallest angle of rotation in any particular pattern.

2.3 Vertical Reflection

If we were to have a vertical line at any point on our image and when we flip all the points from whatever side they are on to the other then we have vertically flipped our figure. If this action preserves distance and symmetry, then there is a vertical reflection symmetry. In each groups construction we will denote a vertical reflection with v.

2.4 Horizontal Reflection

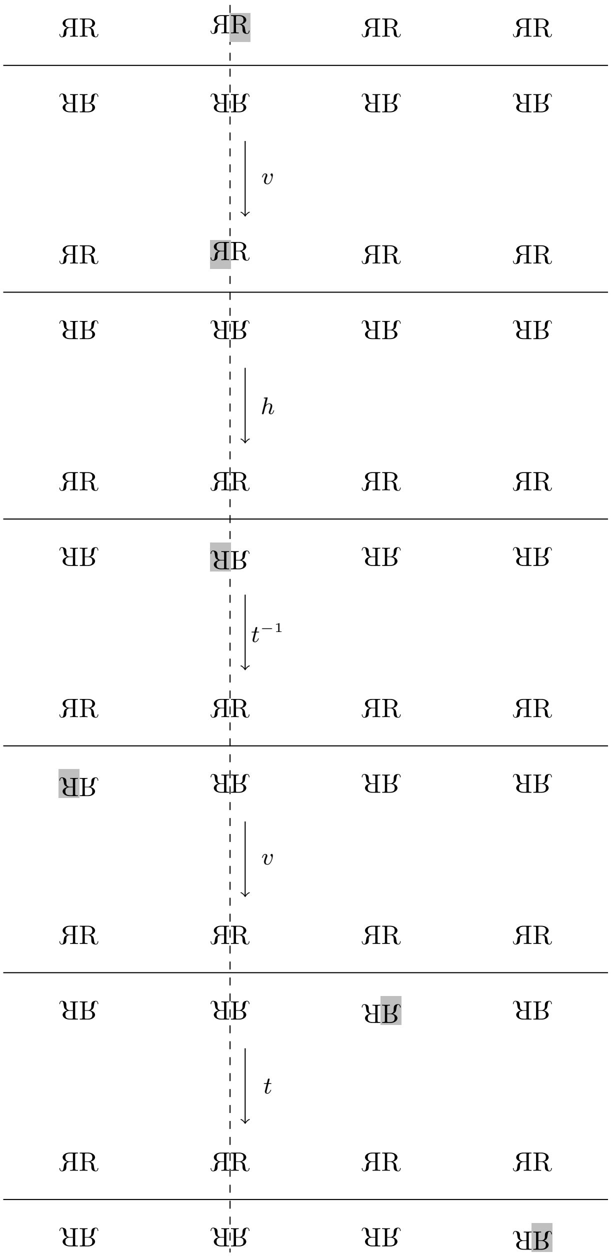
Similar to a vertical reflection if we were to flip all the points from the top to the bottom of a horizontal line and vice verse, then we have horizontally flipped our image. If this action preserves distance and symmetry, then there is a horizontal reflection symmetry. In each group construction we will denote a horizontal reflection with h.

2.5 Glide Reflection

For glide reflections we have a combination of a translation and a horizontal reflection. The motion takes a figure and shifts it half of a translation and then flips the figure across a horizontal axis. In our group constructions we will denote a glide reflection with g. As a side note, if we have a reflection on the same axis as a glide reflection, we will refer to the glide reflection as a trivial one as it is simply a composition of the reflection with a translation. For simplicity sake, we will only acknowledge nontrivial glide reflections in all constructions going forward.

2.6 Motions in Action

To see how to perform the motions of a given group element we will look at a quick, slightly complex example, Suppose we look at a frieze pattern which has both horizontal and vertical reflection, we will see this is our seventh frieze pattern later on. An element of this group will simply be any composition of t, v, or h, suppose we look at $x = tvt^{-1}hv$, it is important to note since these group elements are compositions of isometries then we will work from right to left when looking at the actual movement of our figure. We will now show this complex composition can be simplified into a cleaner expression to make it easier to comprehend at first glance.



Thus we see our x value simplifies to $x = tvt^{-1}hv = t^2h$

3 Normality of Translations

In order to easily apply the semidirect product to see isomorphism classes for each of the frieze patterns, we will shows translations are a normal subgroup of every pattern. Suppose we look at the following:

$$mt^n m^{-1}$$

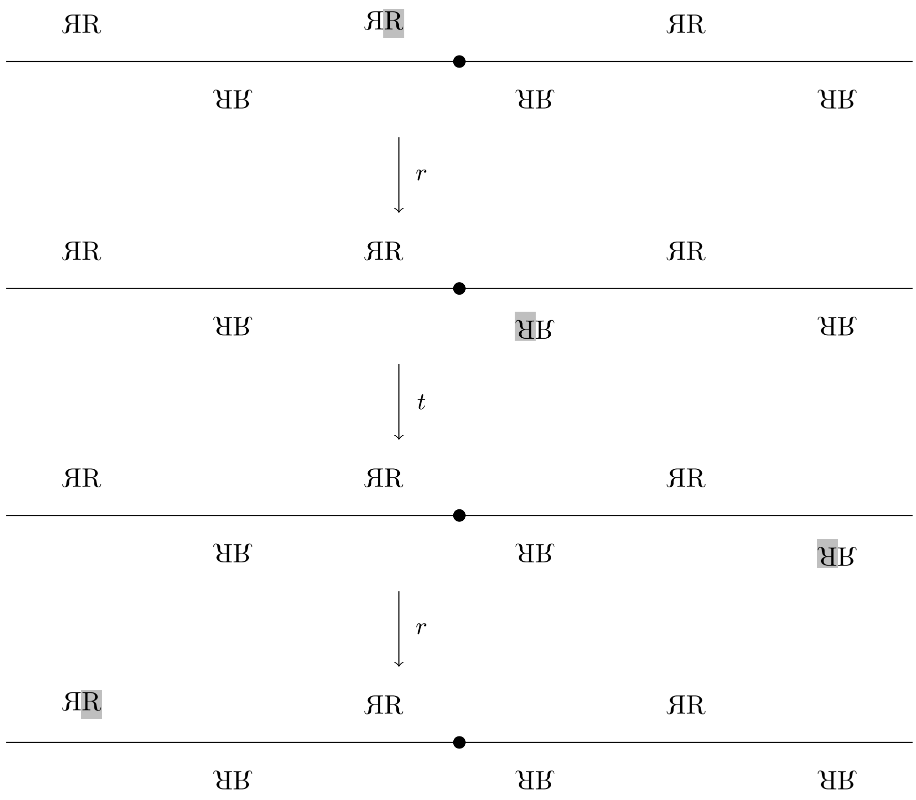
where m is any of the five motions and t^n is any translation. If we were to extend our term out, we see this is the same as $(mtm^{-1})^n$, so what we really need to show is that mtm^{-1} is always a translation. It is also important to note that our argument will be the same for each of the abelian (glide reflections and horizontal reflections) and non-abelian (vertical reflections and rotations) motions. We will present our theorem here and present a justification as we see each motion composed with only translations to see the normality of translations with each motion.

Theorem 3.1. *For all motions (m) and a translation (t):*

$$mtm^{-1} \in T.$$

Proof. **Case 1:** In our first case our m will be a translation, thus we will be looking at $t^n t t_1^{-n}$, which is simply the product of three translations. Since translations are a group this will be another translation.

Case 2: Our next possibility is when m is a rotation, which in the case of our frieze groups we see that $r = r^{-1}$ as $r = \pi$. Now we will show by applying the composition rtr to a group with a rotational symmetry that this will be a translation.



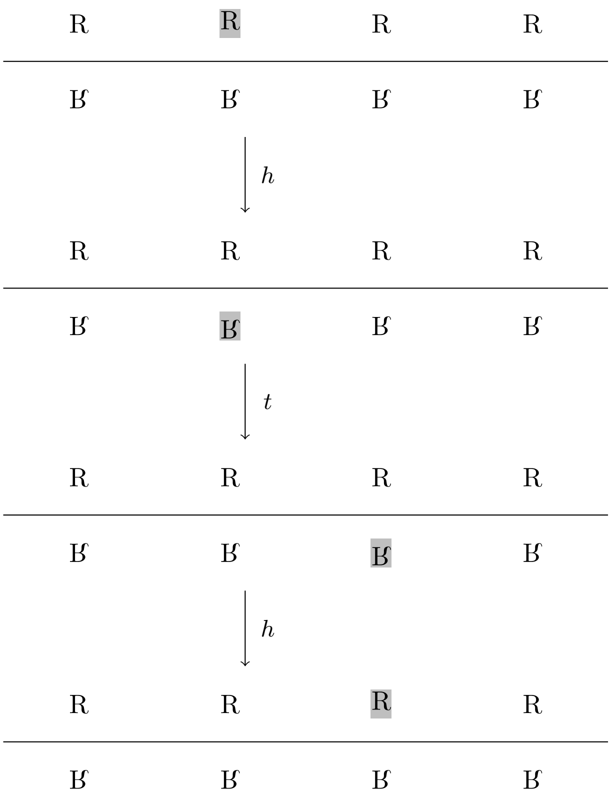
As illustrated by the diagram above, the element rtr will result in an inverse translation (t^{-1}) i.e. an element of T.

Case 3: The third possibility for our m is a vertical reflection , v. Vertical reflections are another motion we see in our generation is non-abelian, and it follows in the same way as above that vtv will be t^{-1}, implying vtv is an element of T.

Case 4: In our fourth possibility we will look the case where m is a horizontal reflection. We notice $h^2 = 1$ which implies $h = h^{-1}$. Horizontal reflections and translations commute, so we observe that

$$hth^{-1} = hth = h^2 t = t.$$

Thus hth is an element of T. The following diagram will demonstrate this claim further.



Case 5: For our final case we will look at when m is a glide reflection. In this case it follows directly from the same argument as our horizontal reflection as it is also commutes with translations, so what we have is:

$$gtg^{-1} = gg^{-1}t = t.$$

This is our final motion, so we have now verified that indeed for any motion m we have $mtm^{-1} \in T$. ∎

4 Classification and Construction of Frieze Groups

For the examples of each pattern that follows we will be using figures composed of R's, as the letter has no inner symmetry. The lack of symmetry allows us to simply orient multiple R's to create whatever symmetry we desire. For example, we can create a vertical reflection symmetry by orienting two R's back to back as ЯR [3]. We will also be applying our semidirect product to classify each pattern into an isomorphism class. We noted before that our translations will be a normal subgroup of each frieze pattern. This fact will allow us to conclude the group of translations will always be the H group we mentioned in our discussion of the semidirect product. Since our group of translations is isomorphic to $\mathbb{Z}$ by definition, then we also know our H is a cyclic group. When we mod out our translations we will be left with a remaining subgroup of $\mathbb{Z}_2$ as our K in nearly every group, and the distinguishing factor in each group isomorphism class is the ϕ function we use between our K and $\mathrm{Aut}(H)$.

4.1 Pattern 1

Our first frieze group pattern is one with no symmetry aside from translations. As stated before, every group pattern will contain transformations, and this motion alone is shown below with its generation.

t^{-1}	e	t	t^2
R	R	R	R

$$\langle t \rangle$$

This frieze group is isomorphic to $\mathbb{Z}$ as it consists of solely translations which by definition are isomorphic to $\mathbb{Z}$.

4.2 Pattern 2

The second frieze pattern will be constructed with translations and glide reflections. Even though glide reflections emulate translations, as $g^2 = t$ we still recognize the groups as being generated from the two motions together. Below is an example of the pattern as well as the generation of the pattern.

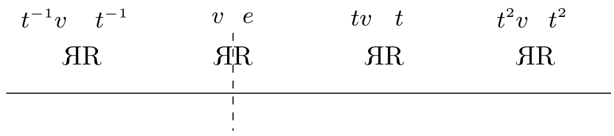
g^{-2}		e		g^2		g^4
R		R		R		R
	ʁ		ʁ		ʁ	
	g^{-1}		g		g^3	

$$\langle t, g | g^2 = t; tg = gt \rangle$$

By our definition of a glide reflection, it is fairly straight forward to see there is a bijection between the glide reflections and translations, namely $\phi(g^n) = t^n$, thus pattern 2 is isomorphic to $\mathbb{Z}$.

4.3 Pattern 3

Our next frieze pattern consists of translations with only vertical reflections as the other possible motion. An example of our group pattern as well as the generator are below:



$$\langle t, v | v^2 = 1; tv = vt^{-1} \rangle$$

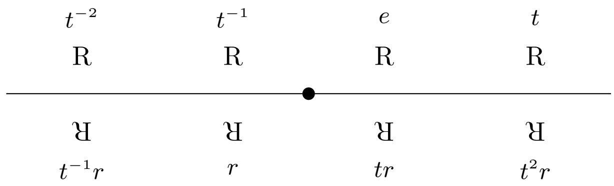
We see above in our generation that this pattern is not abelian, thus it cannot be isomorphic to $\mathbb{Z}$. Now we will start by looking at what we get when we mod out our translations as our H group, this gives us $H = T \cong \mathbb{Z}$ and $K = \{1, v\}$. Since H is normal we know there exists a homomorphism ϕ from K to $\mathrm{Aut}(H)$ and since our G group is non-abelian we know the ϕ function will not be the trivial one sending both elements of K to the identity automorphism. We will now define ϕ as our inversion mapping we described in section 2, which implies for all h in H the following holds:

$$vhv^{-1} = h^{-1}.$$

As we noted in our example of the semidirect product, we have $H = \mathbb{Z}$ and $K = \mathbb{Z}_2$, with our ϕ mapping to inversion, so we see $H \rtimes_\phi K = \mathbb{D}_\infty$. From here we can now conclude this pattern is isomorphic to $\mathbb{D}_\infty$.

4.4 Pattern 4

Our fourth pattern is generated with rotations and translations with an example below:

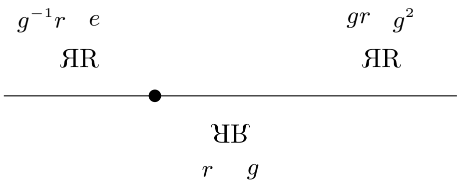


$$\langle t, r | r^2 = 1; rt = t^{-1}r \rangle$$

Similarly to our previous pattern we will once again mod out the translations as our H group giving us $H = T \cong \mathbb{Z}$ and $K = \{1, r\}$. Since this group is also non-commutative we will once again have our ϕ map to inversion. It follows from these conditions that this pattern is also isomorphic to $\mathbb{D}_\infty$.

4.5 Pattern 5

Our next pattern is the first one generated with multiple motions aside from translations. We will generate this group with vertical reflections, rotations, and translations, and we will also notice that vertical reflections are automatically formed from this group. The generations and example is below:

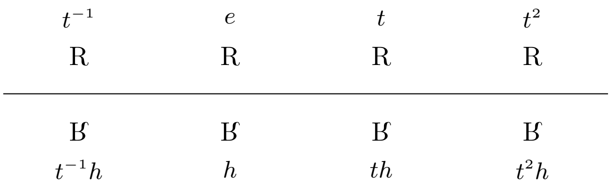


$$\langle t, g, r | g^2 = t; r^2 = 1; tg = gt; rt = t^{-1}r \rangle$$

Since we saw in pattern 2 the group of glide reflections is isomorphic to T then will mod out the glide reflections as our H group leaving us with $K = \{1, r\}$. From here we will make the same conclusion as a ϕ function as we did in pattern 4, and it follows that our semidirect product will result in a G group of $\mathbb{D}_\infty$. This implies we will once again have a pattern which is isomorphic to $\mathbb{D}_\infty$.

4.6 Pattern 6

The sixth pattern consists of horizontal reflections and translations. The following is an example with the generation as well:



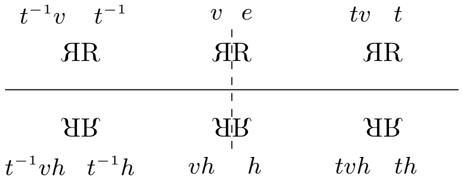
$$\langle t, h | h^2 = 1; ht = th \rangle$$

As we have seen before we will once again have $H = T$ and $K = \{1, h\}$, but this time we notice that our group in abelian, so we will actually have each element of K map to the identity automorphism. This is a trivial semidirect product as the mapping results in the same G group as we would have if we took the direct product. Thus since we have one element in our group that is of order 2 we have the following isomorphism class:

$$H \rtimes_\phi K = \mathbb{Z} \oplus \mathbb{Z}_2.$$

4.7 Pattern 7

Our final frieze pattern consists of horizontal and vertical reflections with translations. Similar to pattern 5 this pattern also automatically fulfills another motion namely rotations. An example of the pattern and its generation follow below:



$$\langle t, h, v | v^2 = 1; h^2 = 1; tv = vt^{-1}; ht = th; hv = vh \rangle$$

This group is slightly different from each of the groups before as we will now mod out our direct product of $\mathbb{Z} \oplus \mathbb{Z}_2$ we get from our translations and horizontal reflections as our H group. This means we have a slightly new automorphism in our H group. Since the inverse of any element in $\mathbb{Z}_2$ is itself we know there is only one automorphism in $\mathbb{Z}_2$ and we still have our normal automorphisms in $\mathbb{Z}$, so our automorphisms on H are as follows:

$$\begin{cases} (x, y) \to (x, y) \\ (x, y) \to (x^{-1}, y). \end{cases}$$

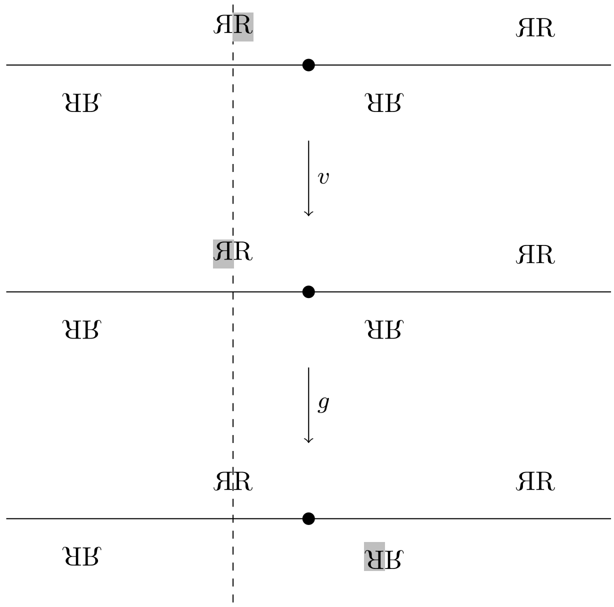
We will also notice again our vertical reflections ($K = \mathbb{Z}_2$) are non-commutative so our ϕ function will unsurprisingly map to inversion. We can now construct our semidirect product which gives us:

$$G = H \rtimes_\phi K = \mathbb{Z} \oplus \mathbb{Z}_2 \rtimes_\phi \mathbb{Z}_2 = \mathbb{D}_\infty \oplus \mathbb{Z}_2.$$

This shows the isomorphism class for our last pattern is $\mathbb{D}_\infty \oplus \mathbb{Z}_2$.

5 Why Only 7 Frieze Patterns

By definition of a frieze pattern we know translations are involved in each group. We also know every motion g, v, r, and h composed with translations is represented by patterns 2, 3, 4, and 6 respectively. First we will note our remaining two groups are pattern 5 and 7 which are constructed by $\langle t, g, r \rangle$ and $\langle t, h, v \rangle$ respectively. From here we know there are only four remaining possibilities for compositions since we noted before that h and g will not be in the same construction as this results in a trivial glide reflection. The possible constructions are the following: $\langle t, g, v \rangle$, $\langle t, v, r \rangle$, $\langle t, h, r \rangle$, and $\langle t, h, v, r \rangle$. We will show these constructions are isomorphic to the seven groups we have already shown constructions for. In order to prove the isomorphisms we will show the unaccounted for motions can be constructed through compositions of motions in the 7 groups we already have. Suppose we look at the following element of $\langle t, g, v \rangle$: $x = gv$



What we see from the example above is that since $r = gv$, then our group $\langle t, g, v\rangle$ is isomorphic to $\langle t, g, r\rangle$ which is our fifth frieze pattern. In the same way as before we can see $g = rv$ which implies another isomorphism of $\langle t, v, r\rangle \cong \langle t, g, r\rangle$. We will use the same argument of showing each motion can be written as a composition of different motions to show isomorphic generators. The element we will notice from $\langle t, h, r\rangle$ is $x = t^{-1}hr$, this is seen to simply be a vertical reflection, this implies $\langle t, h, r\rangle \cong \langle t, h, v\rangle$. If we switch around some of the motions from the element we looked at we see $r = htv$ which implies that in the group $\langle t, h, v, r\rangle$ the r element is trivial thus it need not be in our group construction. This shows our final possibility of a group represented by $\langle t, h, v, r\rangle$ is isomorphic to $\langle t, h, v\rangle$.

This allows us to conclude that our seven frieze patterns we mentioned above represent every possible generation of a frieze group. The groups which are not directly written in our pattern generations are isomorphic to a given pattern through composition of the missing motion using the ones in our generator. It is truly a fascinating result that the number of possible generations of these patterns is a number as unintuitive as 7. This just shows how interesting the world of mathematics is. We might expect to see a pattern as we continue increasing our dimension, but as we will see in the next section, two dimensional plane symmetries have 17 patterns, while three dimensions increases to a massive 230 patterns. Based on the knowledge of available patterns for groups up to three dimensions, there is clearly no pattern to the sequence of $\{7, 17, 230\}$, and it is hard to imagine more dimensions would result in a pattern being revealed.

6 Introduction to Wallpaper Groups

As an extension of frieze patterns we can add another dimension to come up with another group formally called "crystallographic" groups, but are frequently referred to as "wallpaper" groups. We will again recall the definition of a wallpaper group is a plane symmetry group whose translations are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The motions on wallpaper groups are slightly different than the frieze group motions. Instead of simply having vertical and horizontal reflection axes, there are multiple different possibilities depending on the orientation of the figure. In addition to having translations in multiple directions as well as many possible reflection axes our glide reflections are not only restricted to a horizontal reflection axis, they can go along any axis as long as it is not a reflection axis as these are still trivial glide reflections. Finally, as we stated before, the rotational symmetries on wallpaper groups are not limited to only a rotation of $180°$, rather there is much more flexibility providing many different group structure and ultimately influencing the possibilities of different reflection axes as well.

6.1 Only 5 Rotations

When visualizing a rotational symmetry in our plane, we know we have figure being translated in two distinct directions, so where the symmetry must first occur is in our figure being translated. From here we can think of general geometric shapes to draw around our figure to test the rotational symmetry. The shapes would have to be regular polygons as they have the group of rotations as a subgroup to their symmetries. Now we can begin to look at the different possible rotational symmetries on wallpaper groups as they are usually identified by their rotational symmetry. It is also important to note they are identified by the lowest degree of rotational symmetry for a given group.

There are only 5 possible rotational symmetries: $60°$, $90°$, $120°$, $180°$, or no rotational symmetry. To determine why these are the only rotations we will first find the regular polygons with internal angles which divide 360, as they will be the possible rotationally symmetrical angles. There are only two capabilities of rotations not represented by these polygons and they are fairly intuitive ones. The first is one in which there is no rotational symmetry, as we cannot have a regular polygon with internal angles of 0 or 360. The other way we can have rotational symmetry not representable with the internal angles is one of 180 as this would result in no angle the edge would just be continuing in a linear direction. Thus we will start looking at the internal angles of regular polygons which we know can be represented with the following formula:

$$\theta = \frac{n-2}{n}(180).$$

Since we know the angle will have to divide 360 and our n has to be a minimum of 3, then we can very quickly find out what possible angles there are. First we will note $n = 3$ is our first possible n and it results in an angle of 60, which divides 360, thus a rotational symmetry of 60 is possible. Now we will notice the only other angles that divide 360 that are greater than 60 are: 72, 90, and 120. Now we will look at a table of the first 5 n values and their resulting interior angles:

n	*angle*
3	60
4	90
5	108
6	120
7	$128\frac{4}{7}$

Using these values, we know the angle with keep growing and approach 180, thus we see the only n values which result in a number that divides 360 are 3,4, and 6. Thus we can conclude only 5 rotational symmetries exist: 60°, 90°, 120°, 180°, or no rotational symmetry. Now that we have set the classifications of rotational symmetries, we will proceed to look at the reflections and glide reflections within each of these classes as they are the only other possible motions.

7 Description of 17 Wallpaper Groups

We will now use our knowledge about the possible rotations in wallpaper groups and look at the other motions we can apply to each of these classes. The notation of each group is also important to be aware of, as the names tell us about the structure of a group. In each pattern name, the first number (if there is one) refers to the degree of the rotational symmetry. In some cases the degree is not mentioned, as either there is no rotational symmetry or the other symmetries naturally fulfill a rotational symmetry of 180°, so the description is trivial. The p used to start almost every pattern is fairly unimportant, it is simply a standard letter to start group names with. There are two groups which start with a c instead of a p, and this is important as they contain a non-trivial glide reflection in their construction. The final two letters we see are m and g. With g they very nicely abbreviate what they represent which is a glide reflection axis, and if multiple g's are used there are multiple **different** axes. For m it does not naturally follow what motion it would be, but the represent a plain reflection axis. Similarly to g if there are multiple m's, then there are multiple **different** reflection axes. We will note an important fact that the number of m's and g's do not always directly correspond to the number of respective axes when dealing with the groups of rotational order greater than 2. We will now do a brief investigation into each class of rotational symmetry and break down each of the groups within their respective class.

7.1 No Rotational Symmetry

When we start to think about reflection axes we can see from simple geometry that two different directions of reflection axes results in having some sort of rotational symmetry as well. Since the groups we are considering here have no rotational symmetry we will first start by asking if there is a reflection axis. If there is no reflection axis then we will see if there is a glide reflection, if there is them we call our group pg if not it is simply called $p1$. If there is reflection axis then we will also ask about whether this is a non-trivial glide reflection, if so we have the group cm, if not we have pm which is our final group with no rotations.

7.2 180° Rotational Symmetry

With 180° rotational symmetry we can have reflections in multiple directions but since there is a rotational symmetry of 180° we can conclude from a geometric sense that they must be perpendicular. Similarly we will start by asking whether or not there are any reflection axes. If not, we go to glide reflections and the group with glide reflections is called *pgg* as there will be two glide reflection axes, and if not then we are left with our simplest group in this class called $p2$. Moving back to the groups with reflection axes we will now ask whether there are reflections in two directions, if not then we have a group with a reflection axis and a perpendicular glide reflection axis which we call *pmg*. If we do have reflection axes in two directions then the next question we ask is whether the rotation centers are on axes of reflection, if so we are left the group *pmm* if not we have the last group with rotational symmetry of 180° which is *cmm*.

7.3 120° Rotational Symmetry

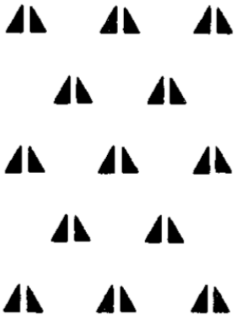
As we have done in each of the previous classifications we start by asking if there are reflection axes, and with this rotational symmetry it is simpler to imagine symmetries within a regular triangle. With a regular triangle there only three possible reflection axes, and if we imagine the center of this triangle also having symmetry then we realize each of the reflection axes will hold symmetry if one does. So with this knowledge when we think about having reflections, if we do not then there cannot be glide reflections which means we are left with the group of $p3$ which only has rotational symmetry. If there are reflection axes, then we figure out whether each of the rotational vertices are on a reflection axis or not. When the vertices are all on reflection axes then we have our group $p3m1$, and if not then we have the final group in this classification of $p31m$.

7.4 90° Rotational Symmetry

Once again we will ask whether or not the pattern we are considering has reflection axes and similarly to the previous group of classifications, if there are not reflections, then we are left with our basic group of $p4$ which only has rotational symmetry. Of the groups with reflection axes, there are only two, one with which has reflection axes in 4 directions and the other only contains reflections in 2 directions. The group with 4 reflection axes is called $p4m$ and the other is referred to as $p4g$ as our reflection axes are usually two glide reflection axes and even if they are not we can think of them as two trivial glide reflection axes.

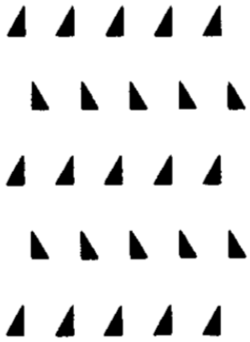
7.5 60° Rotational Symmetry

The high order of rotation for this group limits the ability for other symmetries, in particular we only have two groups here, the typical group which only has rotations ($p6$) and a group with reflections ($p6m$). This is the extent of the wallpaper description we will do as the classification into isomorphism classes is much more complex than in our frieze groups. This is in large part to the high orders of rotations we have with increasing the dimensions. The big problem which separates the wallpaper groups from frieze is that when we mod out translations we are left which a much more complex group than just $\mathbb{Z}_2$ in most cases. This problem prevents us from simply applying the semidirect product between our two subgroups.



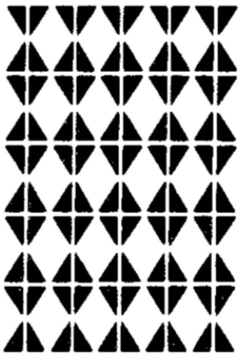
(a) *cm*

(b) *pm*

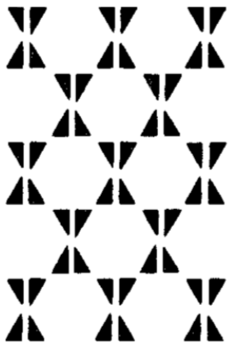


(c) *pg*

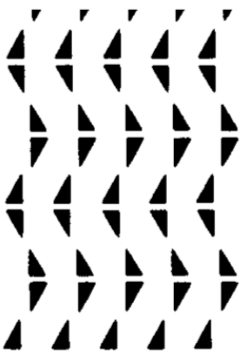
(d) *p*1



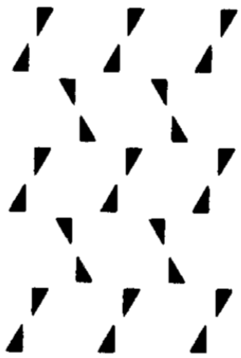
(e) *pmm*



(f) *cmm*

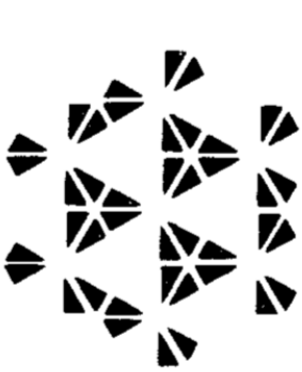


(g) *pmg*

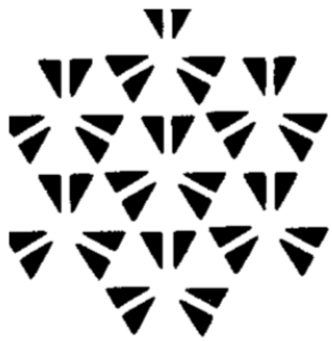


(h) *pgg*

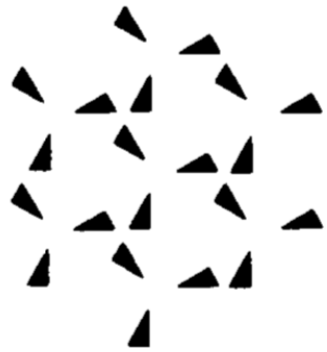
(i) *p*2



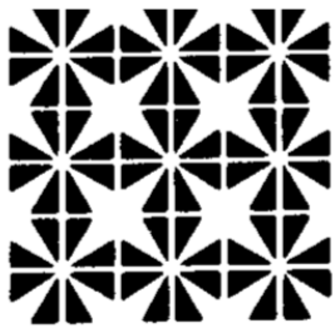
(a) $p3m1$



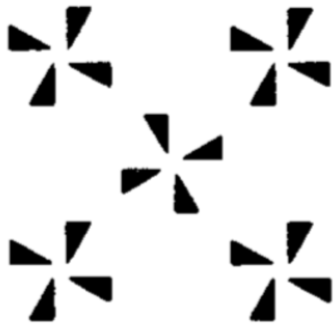
(b) $p31m$



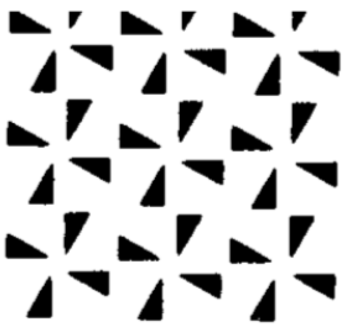
(c) $p3$



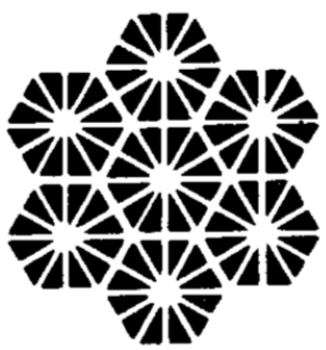
(d) $p4m$



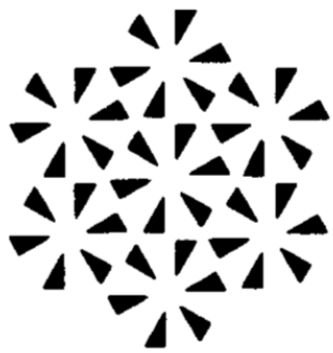
(e) $p4g$



(f) $p4$



(g) $p6m$



(h) $p6$

8 Conclusion

After looking at both frieze and wallpaper patterns, there are clearly some unique applications to the images available to form. Once we are privy of the various figures, it is hard to not notice them wherever you go. There is much more work to be done in the classification side of wallpaper groups, in particular finding the isomorphism classes of each pattern. Since we will not be left with simply a $\mathbb{Z}_2$ group when we mod out translations, then we are not able to apply a ϕ function for a semidirect product as we did with frieze groups as there are still only two automorphisms on our translations. In order to further classify the wallpaper groups we will need a better knowledge of constructions of large (infinite) non-abelian groups. The semidirect product only gives us one type of construction for non-abelian groups, but it is not perfect as it is not applicable to all groups. Even though we do not have specific groups to refer to the patterns as, there are still concrete descriptions that give us a very particular and unique set of 17 possible patterns. This is one of the beauties of math though, examination of new fascinating ideas drives us to look further and find a way to a solution.

9 Acknowledgments

I would like to acknowledge prof. Barry Balof, who introduced me to the idea of frieze patterns as well as oversaw my progression on the topic through the semester. Recognition to Nathaniel Larson is also necessary as he helped edit my paper through many drafts, giving me very helpful advice especially the writing aspect of this project.

References

[1] T. E. o. E. Britannica, "Frieze," May 2018.

[2] D. S. Dummit and R. M. Foote, *Abstract Algebra.* John Wiley & Sons, Inc., 2004.

[3] J. A. Gallian, *Abstract Algebra.* Richard Stratton, 2010.