3. (a) [9 marks] Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset $U$ of $\mathbb{C}$ and let $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$.

State and prove the *Cauchy–Riemann equations* satisfied by $u$ and $v$.

Consider the continuous function $\sqrt{z}$ on the cut plane $\mathbb{C}\backslash(-\infty, 0]$ whose real and imaginary parts satisfy

$$u^2 - v^2 = x, \qquad 2uv = y.$$

Verify explicitly the Cauchy–Riemann equations in this case.

(b) [8 marks]

(i) Define the holomorphic branch $L(z)$ of $\log z$ on the cut-plane $\mathbb{C}\backslash R_\alpha$, such that $L(1) = 0$, where the ray $R_\alpha$ is given by $R_\alpha = \{z \in \mathbb{C} : z = re^{i\alpha}, r \in [0, \infty)\}$ with $2\pi > \alpha > 0$. Assume $\alpha \neq \pi/2$ and compute $i^i$ using this branch.

(ii) Define a holomorphic branch of $f(z) = \log(z^2 - 1)$ on the cut-plane

$$\mathbb{C}\backslash\{(-\infty, -1] \cup [1, \infty)\}.$$

Show that the branch is single valued as we cross the real axis away from the cut.

(iii) Define a holomorphic branch of $f(z) = \log \frac{z-1}{z+1}$ on the cut-plane $\mathbb{C}\backslash[-1, 1]$. Show that the branch is single valued as we cross the real axis away from the cut.

(c) [8 marks] Explain why the following sequence of functions

$$f_n(z) = \sum_{k=-n}^{n} \frac{(-1)^k}{(z+k)^2}$$

converges to a holomorphic function $f(z)$ on $\mathbb{C}\backslash\mathbb{Z}$ as $n \to \infty$. Explain why $f(z)$ is periodic and then find a closed form expression for it. Use this expression to compute

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.$$

# 1 Complex Analysis (problems 3,4,5) - Model solutions

1. (Solution)

(a)[B] The Cauchy-Riemann equations state $u_x = v_y$ and $u_y = -v_x$. [2 marks]

We can take $f'(z)$ in two different ways, and they should agree. Introducing $z = x + iy$ we can compute

$$\partial_x f(z_0) = \lim_{t\to 0} \frac{f(z_0+t) - f(z_0)}{t} = f'(z_0) \tag{1}$$

$$\partial_y f(z_0) = \lim_{t\to 0} \frac{f(z_0+it) - f(z_0)}{t} = i \lim_{t\to 0} \frac{f(z_0+it) - f(z_0)}{it} = i f'(z_0) \tag{2}$$

hence $\partial_y f(z_0) = i\partial_x f(z_0)$. Writing $f(z) = u(z) + iv(z)$ and decomposing into real and imaginary parts, the Cauchy-Riemann equations follow. [4 marks]

[B/S] The real and imaginary part of the square root satisfy $u^2 - v^2 = x$ and $2uv = y$. Hence

$$2(uu_x - vv_x) = 1, \quad 2(uu_y - vv_y) = 0, \quad 2(uv_x + u_x v) = 0, \quad 2(uv_y + u_y v) = 1$$

We can then solve for $u_x, u_y, v_x, v_y$, to find

$$u_x = v_y = \frac{u}{2(u^2+v^2)}, \quad u_y = -v_x = \frac{v}{2(u^2+v^2)}$$

[3 marks]

(b) i.- [S]. Each $z$ on the given cut plane can we written as $z = re^{i\theta}$ with $r > 0$ and $\alpha - 2\pi < \theta < \alpha$. We then define $L(z) = \log r + i\theta$ which indeed satisfies $L(1) = 0$. [1 marks] Now $i^i = e^{iL(i)}$. If $\alpha > \pi/2$ the $L(i) = i\pi/2$. Otherwise $L(i) = -3/2i\pi$. From here we can write $i^i$ for each case. [2 marks]

ii.- [S] Now we define $z - 1 = r_1 e^{i\theta_1}$ and $z + 1 = r_2 e^{i\theta_2}$, where $0 < \theta_1 < 2\pi$ and $-\pi < \theta_2 < \pi$.

$$L(z^2 - 1) = \log(r_1 r_2) + i(\theta_1 + \theta_2)$$

As we cross the real axis along the segment $[-1, 1]$, *e.g.* across $z = 1/2$ or $z = -1/2$, both $\theta_1$ and $\theta_2$ change continuously. [2 marks]

iii.- [S] Again we define $z - 1 = r_1 e^{i\theta_1}$ and $z + 1 = r_2 e^{i\theta_2}$, but now $-\pi < \theta_1, \theta_2 < \pi$. Then

$$L(\frac{z-1}{z+1}) = \log(r_1/r_2) + i(\theta_1 - \theta_2)$$

As we cross the $x > 1$ part of the real axis, both arguments change continuously. As we cross the part $x < -1$, both arguments change by $2\pi$, but this change cancels in the above combination. [3 marks]

(c) [N] The functions $f_n(z)$ are clearly holomorphic in the complex plane except at $2n + 1$ points where the poles are located. Now we can choose a disk with center not among these points, and small enough radius. The M-test says that the series converge uniformly. [2 marks]

The function $f_n(z)$ has double poles at integer values $z = -k, \cdots, k$ with residue $(-1)^k$. In the limit $n \to \infty$ we see the function is invariant under $z \to z + 2$ [1 mark]. A way to find the result is by noting that $f(z) = -g'(z)$, where $g(z)$ has single poles with residue $(-1)^k$. Then by periodicity it follows $g(z) = \pi/\sin(\pi z)$. Taking the derivative it follows [2 marks]

$$f(z) = \frac{\pi^2}{\sin(\pi z)\tan(\pi z)}$$

To solve the last part we compute the expansion of this function around $z = 0$. We get

$$f(z) = \frac{1}{z^2} - \frac{\pi^2}{6} + \cdots$$

The quadratic pole corresponds to the term $k = 0$ in the sum defining $f(z)$ as a sum. The finite part corresponds to twice the sum we are asked to compute. We find

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$$

[3 marks]

2. (Solution)

(a)[B] By Laurent's Theorem there exist unique $c_n \in \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad 0 < |z-a| < r$$

Then we say that $f(z)$ at $a$ has:

.- A removable singularity if $c_n = 0$ for all $n < 0$.

.- A pole of order $n$ if $c_{-n} \neq 0$ and $c_m = 0$ for all $m < -n$.

.- An essential singularity if $c_n \neq 0$ for infinitely many $n < 0$. [4 marks]

An equivalent definition is to say that $f(z)$ has a removable singularity if it is bounded near $z_0$, a pole if $1/f(z)$ has a removable singularity at $z_0$ and an essential singularity it it has an isolated singularity which is neither removable nor a pole.

[B] For the first function we note $\cos z = 0$ at $z = \pi/2 + 2n\pi$ and $z = -\pi/2 + 2n\pi$. Hence at these points we have double poles, except at $z = \pi/2$ where we have a removable singularity. [2 marks]

[B] In the second case we have single poles at all the $n$ roots of unity. In other words $z = e^{i\pi 2k/n}$, where $k = 0, \cdots, n-1$. [2 marks].