Y. David Gau and Lindsay A. Tartre

The Sidesplitting Story of the Midpoint Polygon

Understanding what *area* means and learning ways to calculate the area of various figures are important objectives in geometry. Students as adults will use concepts related to finding areas of polygons in many contexts, such as finding the area of their backyard or knowing how much wallpaper is needed to cover a wall in their dining room. One context for exploring area relationships is comparing the area of a polygon to the area of its associated midpoint polygon, formed by joining the midpoints of consecutive sides of the original polygon. This article describes activities that examine the patterns and relationships between the areas of polygons and those of their associated midpoint polygons for triangles, quadrilaterals, pentagons, and other polygons. We shall also look at the pattern for regular polygons.

TRIANGLES

A corollary to a very old theorem introduced in most high school geometry books and called the *triangle,* or *parallel proportionality, theorem* states as follows:

SIDESPLITTING THEOREM. *A line segment connecting the midpoints of two sides of a triangle is parallel to, and half the length of, the third side.*

In **figure 1,** D and E are the respective midpoints of sides AB and BC in $\triangle ABC$. Because of the theorem, we also know that $\triangle DBE$ is similar to $\triangle ABC$. By connecting D and E to point F, the midpoint of $\overline{AC}$, we get $\triangle DEF$, the midpoint triangle, which is also one of four congruent smaller triangles, each similar to $\triangle ABC$. So the area of $\triangle DEF$ is one-fourth of the area of $\triangle ABC$, regardless of the type of triangle ABC is. This result is used to explain the relationship found for quadrilaterals.

QUADRILATERALS

The next extension often taken by teachers is to investigate a theorem about quadrilaterals that has been attributed to Pierre Varignon (1654–1722) (Coxeter 1969). Varignon's theorem is as follows:

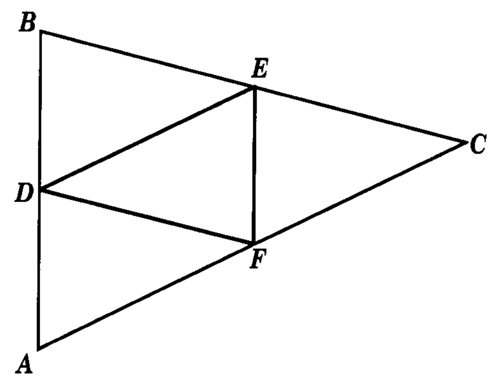


Fig. 1
Midpoint triangle has
one-fourth the area of the original triangle.

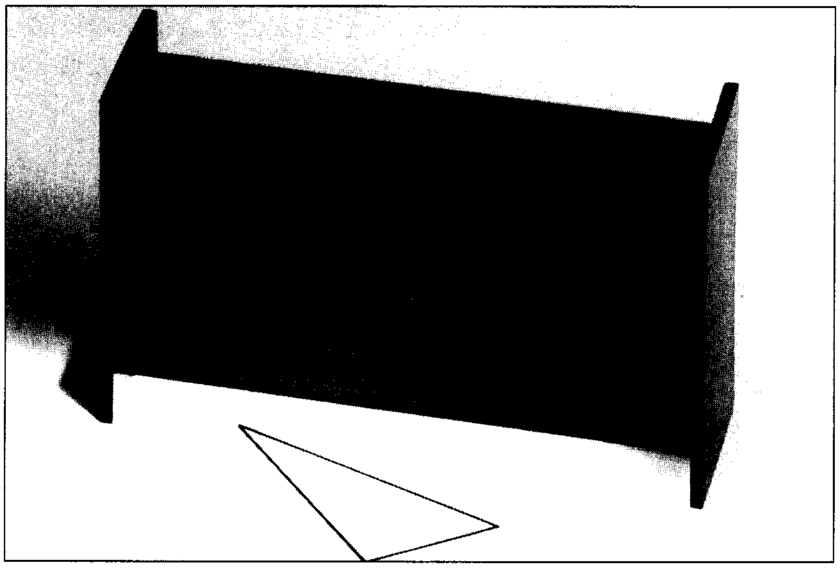
Will the process always work?

VARIGNON'S THEOREM. [*Line segments connecting*] *the midpoints of the sides of a (convex) quadrilateral form a parallelogram that has half the area of the original quadrilateral.*

Some activities relating to this theorem were explored in two issues of *Student Math Notes* (NCTM 1988; White, Olson, and Olson 1989). Varignon's theorem suggests a nice application activity for straightedge and compass or Plexiglas *Mira* or *Reflecta* constructions to find the midpoints of the sides of quadrilaterals (see MIRA Math Co. [1973]). Another possible tool is The Geometric Supposer—Quadrilaterals software from Sunburst (see Houde and Yerushalmy [1986] for activities using this software).

Using any of the foregoing methods, students can generate and investigate patterns and conjec-

David Gau and Lindsay Tartre teach at California State University, Long Beach, Long Beach, CA 90840. Gau teaches a variety of mathematics courses, and his research interests include geometry and the topology of nonsmooth points on curves or surfaces. Tartre is a mathematics educator whose research and teaching activities involve teaching and learning mathematics K-12—particularly geometry, spatial skills, and gender issues.



tures about quadrilaterals even before Varignon's theorem is formally introduced in class. Typically students are asked to identify and verify that the midpoint quadrilateral is always a parallelogram. However, students could also be asked to look for other patterns, or the teacher could ask specific questions. Some possible questions are these: What happens with concave quadrilaterals? What type of parallelogram is formed if the original quadrilateral is a rhombus? Does a relationship exist between the number of lines of symmetry of the original quadrilateral and those of the midpoint quadrilateral? What about rotational symmetry? How do we know that the area of the midpoint quadrilateral is half the area of the original quadrilateral?

Let's start with a special case

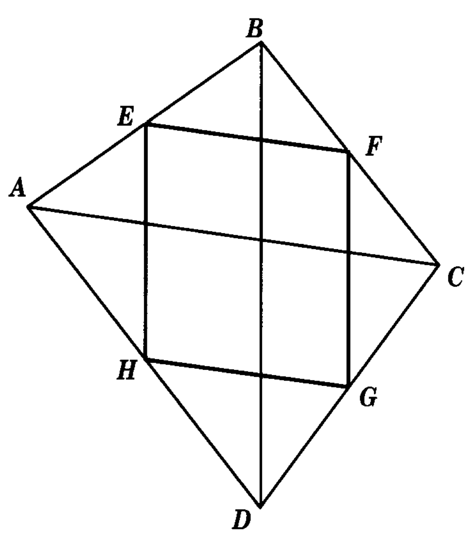


Fig. 2
Midpoint parallelogram has one-half the area of the original quadrilateral.

The part of Varignon's theorem concerning the area of the midpoint quadrilateral has not often been included in high school textbooks. One proof that uses the sidesplitting theorem and the diagonals of the original quadrilateral is shown in **figure 2:**

$$\text{area}(\triangle BEF) = \frac{1}{4}\,\text{area}(\triangle ABC)$$

and

$$\text{area}(\triangle DHG) = \frac{1}{4}\,\text{area}(\triangle ADC).$$

Therefore

$$\text{area}(\triangle BEF) + \text{area}(\triangle DHG) = \frac{1}{4}\,\text{area}(ABCD).$$

Similarly

$$\text{area}(\triangle AEH) + \text{area}(\triangle CGF) = \frac{1}{4}\,\text{area}(ABCD).$$

Thus the complement of the midpoint parallelogram is half the area of the original quadrilateral, and the midpoint parallelogram is half the area of the original quadrilateral.

Paper folding can be used to demonstrate this area relationship in the classroom. Students first draw a quadrilateral using a straightedge and then carefully cut it out **(fig. 3a).** Each side of the quadrilateral is folded in half so that the endpoints meet, and the midpoint is pinched **(fig. 3b).** The paper is then folded along consecutive midpoints creating the midpoint parallelogram and four triangles **(fig. 3c).** After the four triangles are cut out, the students are asked to fit the four triangular pieces on the parallelogram. A similar cut-and-paste activity could be used to demonstrate the midpoint-polygon relationship for triangles.

The teacher can also demonstrate the cut-and-paste process in class. For quadrilaterals, draw a quadrilateral and the midpoint parallelogram on a transparency. Then duplicate the drawing on paper and fold and cut as described previously. Then the triangular pieces of paper and the transparency can be used with the overhead projector to demonstrate the area relationships. Colored transparency material could be cut out instead of paper.

The fact that the outside triangles can indeed be placed intact on the parallelogram **(fig. 3d)** is somewhat surprising to students and leads to several questions:

- Can we justify the process? Will it always work?
- What is the pattern for the placement of those triangles?
- Do all four triangles always meet in one point? If yes, is it unique? Can we identify the point?

We shall begin by examining the special case in which the original quadrilateral is a parallelogram **(fig. 4).** Point I is the intersection of the diagonals of the original quadrilateral, $\overline{AC}$ and $\overline{BD}$. Since $ABCD$ is a parallelogram, I is also the midpoint of

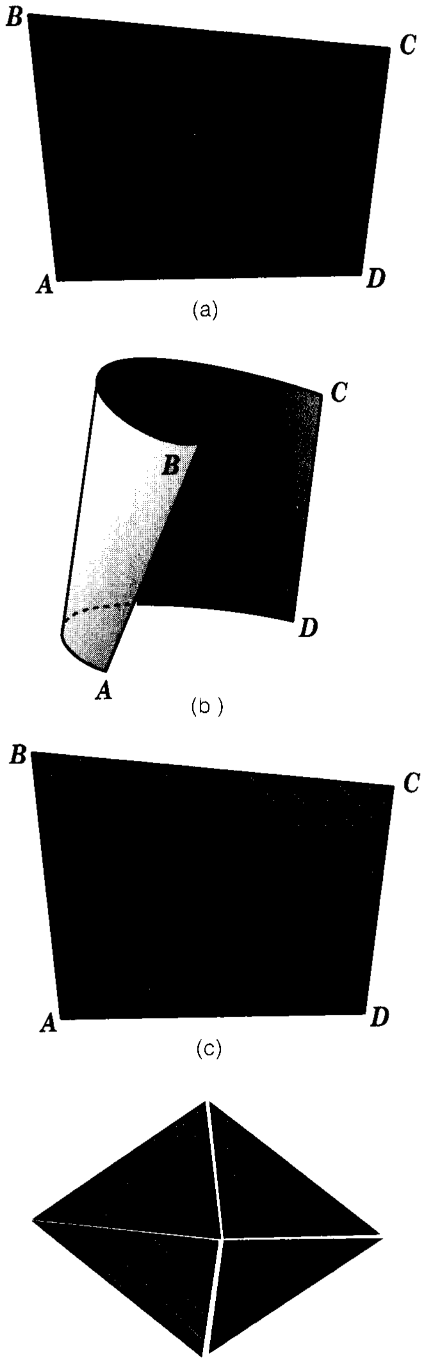


Fig. 3
Paper-folding and cut-and-paste process for midpoint quadrilateral

$\overline{AC}$ and $\overline{BD}$. Connect I to each of the vertices of the midpoint parallelogram to create four triangles. Using the sidesplitting theorem with $\triangle ABD$, we can show that $AEIH$ is a parallelogram, which means that we can rotate $\triangle HAE$ 180 degrees about the midpoint of $\overline{EH}$ and point A will then

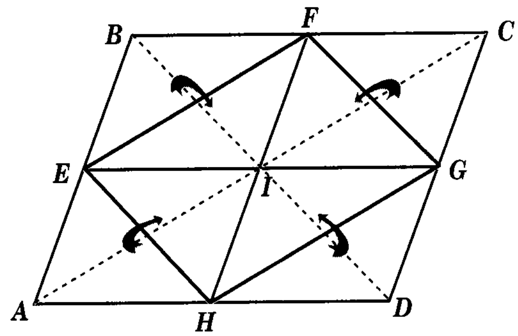


Fig. 4
Midpoint-polygon-area proof for parallelograms

coincide with I. Similarly we can show that $EBFI$, $FCGI$, and $GDHI$ are also parallelograms and that the respective outer triangles $\triangle EBF$, $\triangle FCG$, and $\triangle GDH$ can also be rotated 180 degrees to fit together inside the parallelogram.

What happens with the general quadrilateral? First note that in the general case the intersection of the diagonals, $\overline{AC}$ and $\overline{BD}$, is no longer the midpoint of those diagonals. Let O be the midpoint of one diagonal, say $\overline{AC}$ (**fig. 5**). Then, as in the case of the parallelogram, we can show that $EBFO$ and $GDHO$ are parallelograms allowing us to rotate $\triangle EBF$ and $\triangle GDH$ 180 degrees about the midpoints of $\overline{EF}$ and $\overline{GH}$, respectively. A surprising aspect of this case is that even though the remaining two triangles cannot be rotated inside, they do move inside by translation. We find that $\triangle EOH \cong \triangle FCG$ and $\triangle AEH \cong \triangle OFG$ by the side-side-side theorem, using the equality of opposite sides of parallelograms. The midpoint of either diagonal could be used for this process. In each case the two triangles that are not intersected by the diagonal chosen are rotated and the other two triangles are translated to the opposite side (see **fig. 6**). This activity could be used as an application for developing the language of transformations. Varignon's

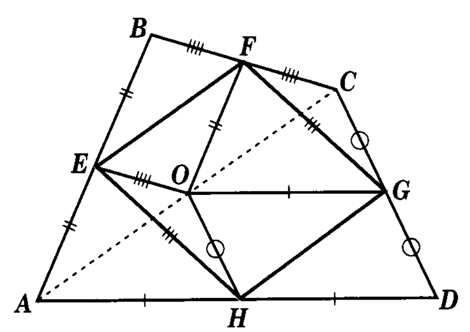


Fig. 5
Midpoint-polygon-area proof for general quadrilateral

Is the area ratio constant for all n-gons?

theorem also holds for concave quadrilaterals (Coxeter and Greitzer 1967) and can be demonstrated by modifying the foregoing cut-and-paste process.

PENTAGONS

We have seen that the ratio of the area of the midpoint triangle to the area of the original triangle, which we shall call the *area ratio,* is always 1/4. We have also seen that the area ratio for quadrilaterals is always 1/2. We next examine whether the area ratio is a constant for n-gons for n greater than 4.

Does every area ratio between 1/2 and 3/4 yield a polygon?

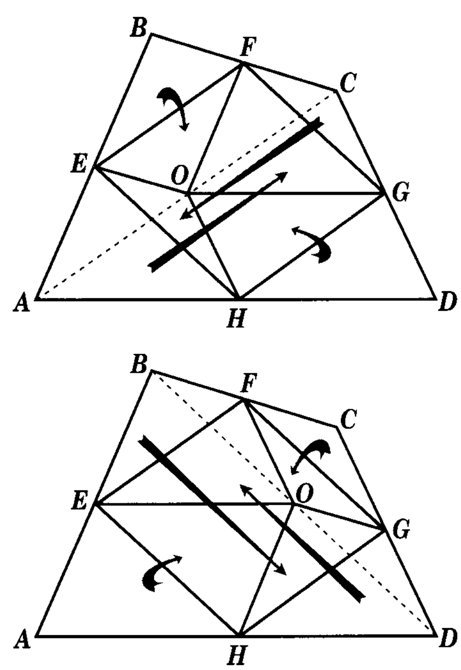


Fig. 6
Transformations to show area relationship

Let us consider pentagons first. One approach for beginning this exploration would be to have students experiment constructing midpoint pentagons using geoboards or square dot paper. (Make sure the students choose the vertices of the original pentagon in such a way that the vertices of the midpoint pentagon are also lattice points.) Then the areas can be found using Pick's theorem. Let I be the number of interior points and B the number of boundary points. Pick's theorem states that the area of a polygon is $I - 1 + B/2$. (See Smith [1990] or Hawkins [1988] for other activities involving Pick's theorem.) Students will discover that the area ratios for pentagons are not constant. They can be challenged to try to obtain the largest and the smallest area ratios. Students could work in cooperative groups for this activity. It turns out that pentagons can have area ratios between 1/2 and 3/4.

AREA RATIO FOR CONVEX PENTAGONS THEOREM. *The area ratio of a convex pentagon is between 1/2 and 3/4. Moreover, any number in that range is the area ratio of some pentagon.*

The proof of this theorem relies on the fact that triangles with equal bases and heights have the same area. For example, in **figure 7** the area of $\triangle DEF$ remains unchanged when F is moved along line AC, which is parallel to $\overline{DE}$. It follows that if two triangles have the same base, the one with the larger height has larger area. First we shall show that area ratios for pentagons are greater than 1/2. Next we shall show that area ratios for pentagons are less than 3/4.

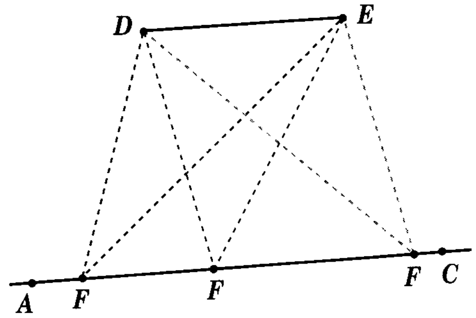
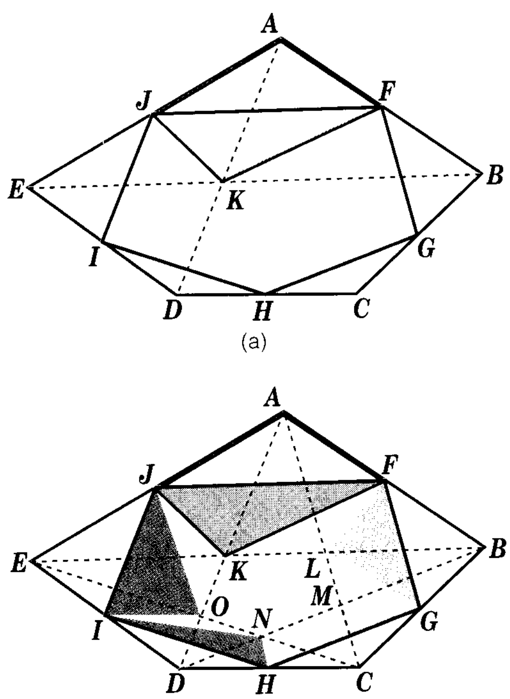


Fig. 7
Constant distance between parallel lines makes triangles with equal area.

Area ratios greater than 1/2. In **figure 8a,** *FGHIJ* is the midpoint polygon for pentagon *ABCDE*. In $\triangle ABE$, if we choose a point K (we have chosen the intersection of $\overline{BE}$ and $\overline{AD}$) on $\overline{BE}$ (which we know is parallel to $\overline{JF}$ from the sidesplitting theorem), then $\triangle JFK$ and $\triangle AFJ$ have the same area. Similarly, in **figure 8b,** $\triangle FGL$ and $\triangle FGB$, $\triangle GHM$ and $\triangle GHC$, $\triangle HIN$ and $\triangle HID$, and $\triangle IJO$ and $\triangle IJE$, by pairs, have the same area. This process is like moving the triangles as we did with quadrilaterals. In this case the triangles, though not congruent, still have the same area. Since pentagon *FGHIJ* is not fully covered when the outside triangles are moved inside, the area of the midpoint pentagon is greater than half the area of the original pentagon *ABCDE.*

Area ratios less than 3/4. The proof that the area ratios for pentagons are less than 3/4 is best understood in terms of what has been called *flexing* (Anderson and Arcidiacono 1989). Suppose that *A, B, C, D,* and *E* are five consecutive vertices of any polygon. To *flex a polygon at the vertex C* means that the polygon is deformed in such a way that *C* moves along the line that passes through *C* and is parallel to $\overline{BD}$ (see **fig. 9**). Notice that the area of the polygon remains unchanged because the area of the moving part, $\triangle BCD$, does not change. We can also flex the polygon at *C* even if *B, C,* and *D* are collinear, and again, the area of the polygon does not change under flexing. We shall consider flexings that make the polygon more degenerate, that is, the polygon will approach becoming a polygon with fewer sides. For example, in **figure 9,** if *ABCDE* are the vertices of a pentagon, then it becomes a quadrilateral when the vertex C is translated along to the intersection of $\overleftrightarrow{CC'}$ and $\overleftrightarrow{AB}$.

What happens to the midpoint polygon when a polygon is flexed? Let *F,G,H,* and *I* be the midpoints of the consecutive sides *AB, BC, CD,* and *DE* of a polygon **(fig. 10).** Suppose E is closer to $\overleftrightarrow{BD}$ than *A* is; then *I* is closer to $\overleftrightarrow{GH}$ than *F* is. Let us flex *C* to *C′*, Since $GH = G'H' = (1/2)BD$, $GG' = HH'$ and area ($\triangle GG'F$) is greater than area ($\triangle HH'I$) (because the height of $\triangle GG'F$ is greater than the height of $\triangle HH'I$), making area ($FG'H'I$) greater than area($FGHI$). Since the area of the original polygon does not change, we have increased the area ratio by flexing. (If we had flexed *C* to *C″*, a similar argument shows that the area ratio



(a)

(b)

Fig. 8
Area ratio greater than 1/2 for a pentagon

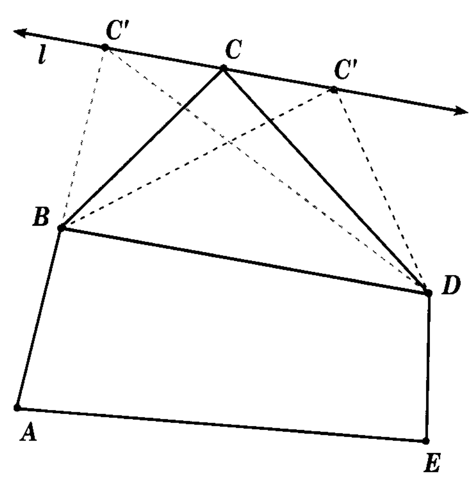


Fig. 9
Flexing process

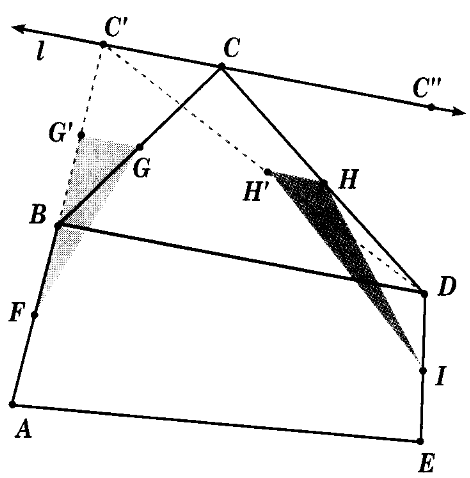


Fig. 10
Flexing to increase area ratio

decreases. If we flex C past C' or C'', then the pentagon becomes nonconvex.) The limit for the area ratio by flexing at any vertex occurs when the vertex becomes collinear with an adjacent side.

We can then show that the area ratio of a convex pentagon is less than 3/4. **Figure 11** shows the process of flexing pentagon $ABCDE$ four times into a limiting triangle. **Figure 11a** shows pentagon $ABCDE$ and its associated midpoint pentagon $FGHIJ$. First we flex B to B' on $\overleftrightarrow{DC}$) because E is closer to $\overline{AC}$ than D is **[fig. 11b]**). Then we flex vertex A to A' on $\overleftrightarrow{DE}$ (because C is closer to $\overline{B'E}$ than D is **[fig. 11c]**). Next we flex C to C' on $\overleftrightarrow{B'A'}$ (because E is closer to $\overline{B'D}$ than A' is **[fig. 11d]**). (Note that C' is also B' and consequently also G''.) Finally we flex vertex E to A' (which is also J'') **(fig. 11e).** In **figure 11e** we see that area ($F'G''H'I'J''$) = 3/4 area ($A'B'D$) as a consequence of the sidesplitting theorem for triangles. Because each step in this specific flexing process increases the area ratio, and because the resulting triangle obviously is not really a pentagon, we know that the area ratio for the original pentagon must be strictly less than 3/4.

Convex pentagons exist whose area ratios are arbitrarily close to 1/2 or 3/4 (see pentagon$_1$ and pentagon$_2$ in **fig. 12**). By continually deforming pentagon$_1$ into pentagon$_2$, we obtain a family of convex pentagons whose area ratios range between the area ratios of pentagon$_1$ and pentagon$_2$. It follows that any number between 1/2 and 3/4 is the area ratio of some convex pentagon.

OTHER POLYGONS

What about hexagons and other n-gons? For the upper bound, the arguments presented previously still apply, but the degenerate polygon (limiting triangle) one obtains at the end of the flexing process will coincide with its own midpoint polygon, making the area ratio 1 (see **fig. 13** for an example). For the lower bound, the argument presented earlier for pentagons can be applied to general n-gons to show that if $n > 4$, the area ratios of n-gons are greater than 1/2. So in general we have the following theorem:

AREA RATIO FOR GENERAL POLYGONS THEOREM. *The*

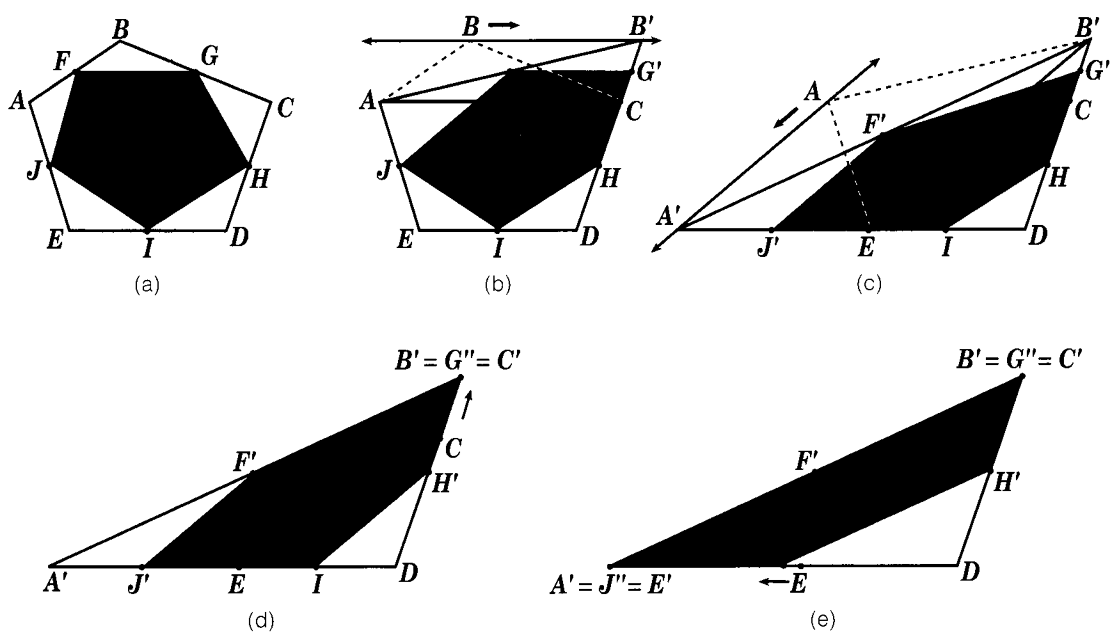
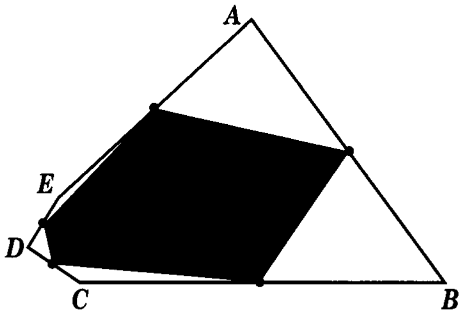
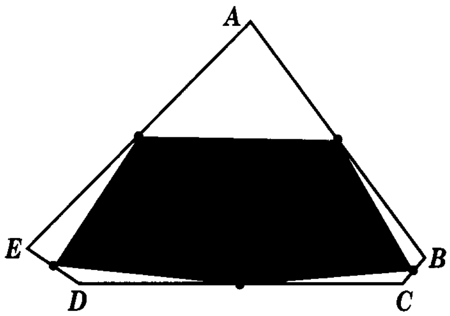


Fig. 11
Flexing process to maximize area ratio for pentagons

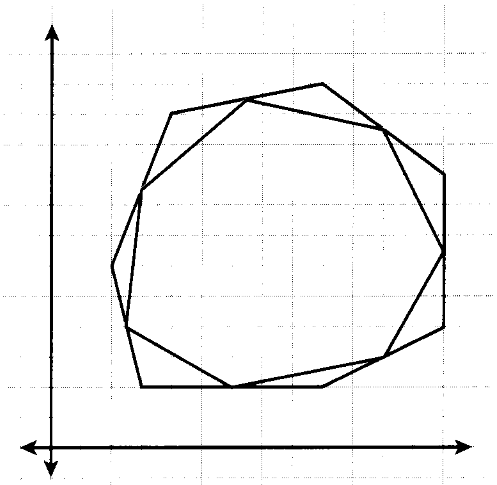


Pentagon$_1$

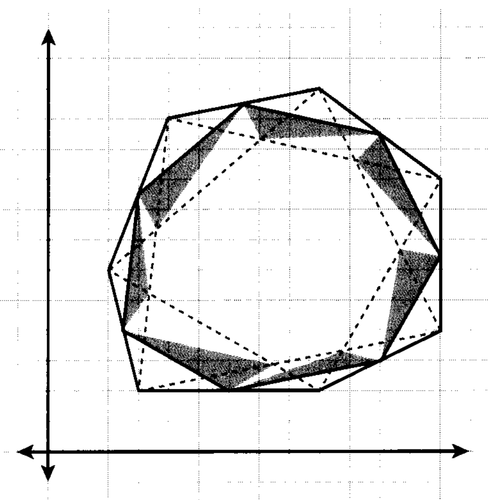


Pentagon$_2$

Fig. 12
Area ratios for pentagons range from 1/2 to 3/4.



(a)



(b)

Fig 14
Activity using coordinates
to explore area ratios for polygons

range of the area ratios for n-gons (n > 5) is between 1/2 and 1.

Students can demonstrate that the area ratios of polygons are greater than 1/2 with graph paper using the midpoint formula and other ideas from coordinate geometry. Suggest that they use lattice points as vertices of the original polygon to make the identification of the midpoints easier. Introduce an origin and axis on the drawing and have students find the midpoints of each side of their *n*-gon (see **fig. 14a**). After constructing their midpoint polygon, students can use the process for the pentagons shown in **figure 8** with their polygons. Students will find it much easier to see if they use one color for the original polygon, a second color for their midpoint polygon, and a third color to draw the diagonals needed. The intersections of diagonals that connect every other point on the original polygon can be used to identify the triangles inside their midpoint polygon that have the same areas as the triangles formed outside the midpoint polygon (see **fig. 14b**). Students will be able to see that the triangles do not cover the entire area of the midpoint polygon. This activity makes continual use of the sidesplitting theorem for triangles and will strengthen students' understanding of the area of a

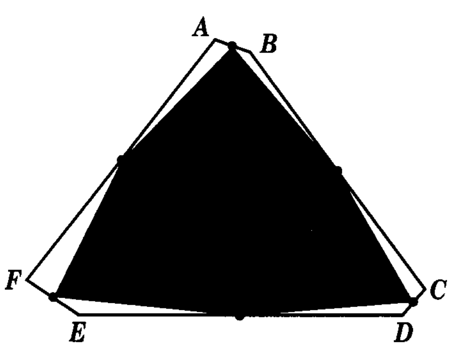


Fig. 13
Maximizing area ratios for hexagons

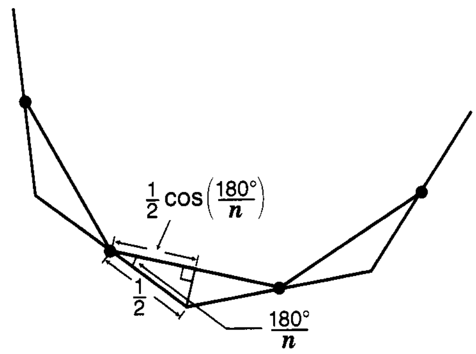


Fig. 15
Area ratio for regular polygons

triangle as well. The dot-paper activities involving Pick's theorem described earlier could be used to demonstrate the area relationships for general polygons as well.

REGULAR POLYGONS

What is the pattern for regular polygons? In this situation, the midpoint polygon is also regular and, therefore, similar to the original polygon. So the area ratio is the square of the ratio between the lengths of the sides of the midpoint polygon and those of the original polygon. From **figure 15** we can see that the latter ratio is $\cos(180°/n)$. So the area ratio for a regular n-gon is $\cos^2(180°/n)$. For $n = 5, 6, 7$, the area ratios are approximately .65, .75, and .81, respectively. Note that the area ratio approaches 1 as n gets larger, which means that as the number of sides (n) increases, the area outside the midpoint polygon gets smaller.

Archimedes used area ratios for regular polygons and their midpoint polygons in a very different context to arrive at an approximation of pi (Dijksterhuis 1987). In **figure 16,** the inscribed hexagon is the midpoint polygon of the circumscribed hexagon. Archimedes reasoned that the area of the circle of radius one unit (π) was larger than the inscribed polygon and smaller than the circumscribed polygon. So he calculated the area of each hexagon and was able to limit the range of the value of pi. He then doubled the number of sides and calculated the area of each polygon again. Through this process of doubling sides, he eventually arrived at a polygon with 96 sides and estimated pi correctly to two decimal places. At the same time he was also demonstrating that as the number of sides of the regular polygon increased, the area of the associated midpoint polygon approached the area of the original polygon.

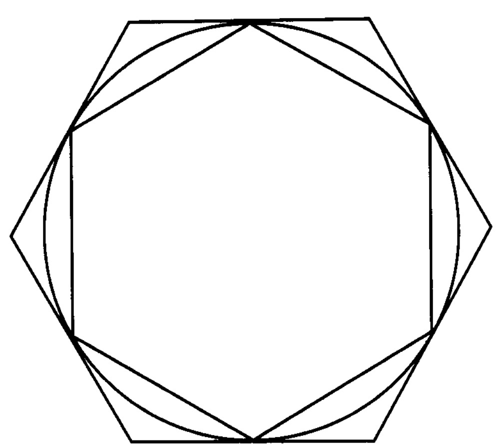


Fig. 16
Using midpoint polygons to approximate pi

BIBLIOGRAPHY

Anderson, David R., and Michael J. Arcidiacono. "Area Ratios of Quadrilaterals." *Mathematics Teacher* 82 (March 1989):176—83.

Coxeter, H. S. M. *Introduction to Geometry.* New York: John Wiley & Sons, 1969.

Coxeter, H. S. M., and S. L. Greitzer. *Geometry Revisited.* Washington, D.C.: Mathematical Association of America, 1967.

Dijksterhuis, E. J. *Archimedes.* Princeton, N.J.: Princeton University Press, 1987.

Hawkins, Vincent J. "Applying Pick's Theorem to Randomized Areas." *Arithmetic Teacher* 36 (October 1988):47–49.

Houde, Richard, and Michael Yerushalmy. "Teaching the Expected with the Geometric Supposer and Meeting the Expected." In *Teaching Mathematics: Strategies That Work*, edited by Mark Driscoll and Jere Confrey, 153–66. Portsmouth, N.H.: Heinemann Books, 1986.

Kulm, Gerald. "Area Ratios in Convex Polygons." *Mathematics Teacher* 67 (May 1974):466–67.

MIRA Math Co. *MIRA Math Activities for Junior High School.* Palo Alto, Calif.: Creative Publications, 1973.

National Council of Teachers of Mathematics. "Midpoint Madness." *Student Math Notes* (January 1988):1–4.

Smith, Lyle R. "Areas and Perimeters of Geoboard Polygons." *Mathematics Teacher* 83 (May 1990):392–98.

White, Gerald, Melfried Olson, and Tanya Olson. "Quadri-gon(e)." *Student Math Notes* (September 1989):1–4.

Wilcock, Douglas. "Start the Year Right—Discover Pick's Theorem." *Mathematics Teacher* 85 (September 1992):424–25.