# Two Identities and their Consequences

Emmanuel Antonio José García
Santo Domingo, Dominican Republic
emmanuelgeogarcia@gmail.com

February 25, 2021

The Heron's formula [1, 4, 7, 14, 15, 17, 18], named after Hero of Alexandria, gives the area of a triangle when the length of all three sides are known. Indian mathematician and astronomer Brahmagupta, in the seventh century, gave the analogous formula for a cyclic convex quadrilateral [6, 10]. In 1842 German mathematician Carl Bretschneider related the area of a general convex quadrilateral to its side lengths and the sum of two opposite angles [5, 8, 11]. Heron's formula is a special case of Brahmagupta's formula for the area of a cyclic quadrilateral. Heron's formula and Brahmagupta's formula are both special cases of Bretschneider's formula for the area of a quadrilateral.

In this note we prove the Heron's formula (although known, see Conway's dicussion in [7]), the Brahmagupta's formula (also known, see [6]) and the formula for the area of a bicentric quadrilateral (possibly new, see [12, 13]), $\sqrt{abcd}$, based on two lesser-known trigonometric formulae [6, 16] involving sine, cosine, the semiperimeter and the side lenghts of a cyclic quadrilateral. Once the two trigonometric formulae have been established (and the necessary adjustments made), the proofs of these area theorems are greatly simplified. Furthermore, we present a generalization of the two aforementioned trigonometric formulae and use it to give an alternative proof of Bretschneider's formula. Since all these area theorems can be derived from this new generalization, the approach presented in this note, unlike others, provides a more holistic view of these theorems. Our main result for a general convex quadrilateral are the identities

$$ad\sin^2\frac{\alpha}{2}+bc\cos^2\frac{\gamma}{2}=(s-a)(s-d)$$

and

$$bc\sin^2\frac{\gamma}{2}+ad\cos^2\frac{\alpha}{2}=(s-b)(s-c),$$

where $a$, $b$, $c$, $d$ are the sides lengths, $s$ is the semiperimeter, and $\alpha$ and $\gamma$ are opposite angles.

We recall a cyclic quadrilateral is a quadrilateral whose vertices all lie on a single circle. Among other characterizations, a convex quadrilateral $ABCD$ is cyclic if and only if its opposite angles are supplementary, that is $\alpha + \gamma = 180°$ [19] (see Figure 1).

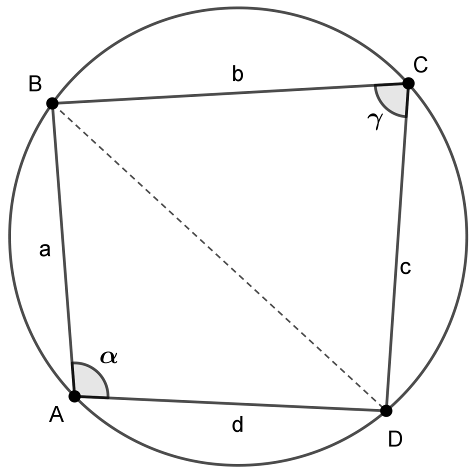


Figure 1: A cyclic quadrilateral $ABCD$.

**Theorem 1.** *Let $ABCD$ be a cyclic convex quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$ and $s = \frac{a+b+c+d}{2}$. If $\angle BAD = \alpha$, then*

$$\sin^2 \frac{\alpha}{2} = \frac{(s-a)(s-d)}{ad+bc} \quad \textit{and} \quad \cos^2 \frac{\alpha}{2} = \frac{(s-b)(s-c)}{ad+bc}. \tag{1}$$

*Proof.* First we will find an expression for $\cos\alpha$ in terms of $a$, $b$, $c$ and $d$. Let $\angle BCD = \gamma$. By the Law of Cosines and keeping in mind that $\alpha$ and $\gamma$ are supplementary, we have

$$a^2 + d^2 - 2ad\cos\alpha = b^2 + c^2 - 2bc\cos(180° - \alpha).$$

Yielding $\cos\alpha = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$. Now, making use of the half angle formula for cosine,

$$\begin{aligned}
\cos^2 \frac{\alpha}{2} &= \frac{2ad + 2bc + a^2 + d^2 - b^2 - c^2}{4(ad+bc)} && (2)\\
&= \frac{(a+d)^2 - (b-c)^2}{4(ad+bc)} \\
&= \frac{(a+b-c+d)(a-b+c+d)}{4(ad+bc)} && (3)\\
&= \frac{1}{ad+bc}\left(\frac{a+b+c+d}{2} - c\right)\left(\frac{a+b+c+d}{2} - b\right)
\end{aligned}$$

$$= \frac{(s-b)(s-c)}{ad+bc}.$$

$\square$

The other formulae can be obtained similarly by replacing $\cos^2\frac{\alpha}{2}$ by $1-\sin^2\frac{\alpha}{2}$ in (2).

In personal email communication with Peter Doyle [7], the renowned mathematician J. H. Conway has given the same proof of Heron's formula that we present here. However, as the aim of this paper is to present these area theorems as mere links of a chain of related theorems from a new standpoint, we deduce (4) in a different way by just setting $c = 0$ in (1).

Here, $\Delta_0$, $\Delta_1$, $\Delta_2$ and $\Delta_3$ stand for the areas of a triangle, a cyclic quadrilateral, a bicentric quadrilateral and a general quadrilateral, respectively.

**Theorem 2** (Heron)**.** *Let a triangle $\triangle ABD$ has sides $AB = a$, $BC = b$ and $AD = d$, then the area is given by the formula*

$$\Delta_0 = \sqrt{s(s-a)(s-b)(s-d)}.$$

*Proof.* For a triangle, if in (1) we assume $c = 0$, then we have the well-known formulae[1]

$$\sin^2\frac{\alpha}{2} = \frac{(s-a)(s-d)}{ad} \quad \textit{and} \quad \cos^2\frac{\alpha}{2} = \frac{s(s-b)}{ad}. \tag{4}$$

Making use of the double-angle identity for sine we have

$$\sin\alpha = 2\sqrt{\frac{s(s-b)}{ad}}\sqrt{\frac{(s-a)(s-d)}{ad}} = 2\frac{\sqrt{s(s-a)(s-b)(s-d)}}{ad}.$$

Since $\Delta_0 = \frac{ad\sin\alpha}{2}$, it follows

$$\Delta_0 = \sqrt{s(s-a)(s-b)(s-d)}.$$

$\square$

As mentioned, the following proof of Brahmagupta's formula is known. However, no further generalizations of (1) are given in [6].

**Theorem 3** (Brahmagupta)**.** *Given an cyclic quadrilateral, $ABCD$, with sides $a$, $b$, $c$, $d$ and semiperimeter, $s$, then its area is given by the formula*

$$\Delta_1 = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

[1] For more implications in a triangle see [9].

*Proof.* Let $\alpha$ and $\gamma$ are two opposite angles. The area of $ABCD$ can be expressed as the sum of the area of $\triangle ABD$ and $\triangle BCD$, which in turn can be written as $\frac{ad\sin\alpha}{2} + \frac{bc\sin\gamma}{2}$. Keeping in mind that $\alpha$ and $\gamma$ are supplementary and applying the formulae in (1) we have

$$\begin{aligned}
\Delta_1 &= \frac{ad\sin\alpha}{2} + \frac{bc\sin(180^\circ - \alpha)}{2} \\
&= \frac{ad\sin\alpha}{2} + \frac{bc\sin\alpha}{2} \\
&= \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}(ad+bc) \\
&= \sqrt{\frac{(s-a)(s-d)}{ad+bc}}\sqrt{\frac{(s-b)(s-c)}{ad+bc}}(ad+bc) \\
&= \sqrt{(s-a)(s-b)(s-c)(s-d)}.
\end{aligned}$$

□

A bicentric quadrilateral is a convex quadrilateral that has both an incircle and a circumcircle (see Figure 2). One characterization states that a convex quadrilateral ABCD with sides a, b, c, d is bicentric if and only if opposite sides satisfy $a + c = b + d$ and its opposite angles are supplementary [19]. Another property of a bicentric quadrilateral is that its area is given by the formula $\sqrt{abcd}$. Six derivations of this formula can be found in [12, 13]. One derivation is to use $a + c = b + d$ in Brahmagupta's Formula. Here we shall give a seventh proof which is independent from Brahmagupta's Formula.

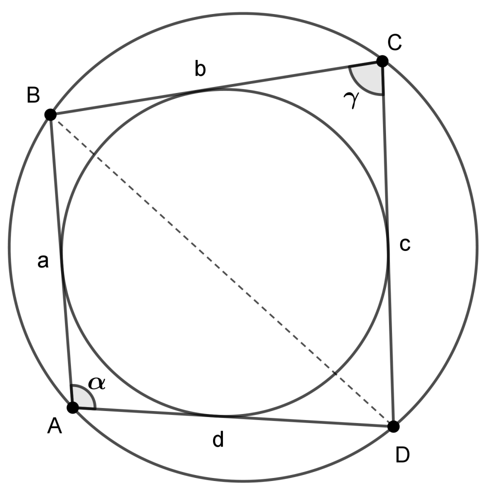


Figure 2: A bicentric quadrilateral $ABCD$.

**Theorem 4.** *Given an bicentric quadrilateral, $ABCD$, with sides $a$, $b$, $c$ and $d$, then its area is given by the formula*

$$\Delta_2 = \sqrt{abcd}.$$

*Proof.* Since $a + c = b + d$ in a bicentric quadrilateral, the formula (3) reduces to

$$cos^2 \frac{\alpha}{2} = \frac{ad}{ad + bc}.$$

Similarly we can get $\sin^2 \frac{\alpha}{2} = \frac{bc}{ad+bc}$. Now, following the same steps as in Brahmagupta's formula

$$\begin{aligned}
\Delta_2 &= \frac{ad \sin \alpha}{2} + \frac{bc \sin (180° - \alpha)}{2} \\
&= \frac{ad \sin \alpha}{2} + \frac{bc \sin \alpha}{2} \\
&= \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} (ad + bc) \\
&= \sqrt{\frac{bc}{ad + bc}} \sqrt{\frac{ad}{ad + bc}} (ad + bc) \\
&= \sqrt{abcd}.
\end{aligned}$$

□

The following theorem generalizes Theorem 1 for a general convex quadrilateral.

**Theorem 5.** *Let $a$, $b$, $c$, $d$ be the sides of a general convex quadrilateral, $s$ is the semiperimeter, and $\alpha$ and $\gamma$ are opposite angles, then*

$$ad \sin^2 \frac{\alpha}{2} + bc \cos^2 \frac{\gamma}{2} = (s - a)(s - d) \tag{5}$$

*and*

$$bc \sin^2 \frac{\gamma}{2} + ad \cos^2 \frac{\alpha}{2} = (s - b)(s - c). \tag{6}$$

*Proof.* By the Law of Cosines,

$$a^2 + d^2 - 2ad \cos \alpha = b^2 + c^2 - 2bc \cos \gamma.$$

Yielding $\cos \alpha = \frac{a^2+d^2-b^2-c^2+2bc \cos \gamma}{2ad}$. Now, making use of the half angle formula for cosine,

$$\cos^2 \frac{\alpha}{2} = \frac{a^2 + d^2 + 2ad - b^2 - c^2 + 2bc \cos \gamma}{4ad} \tag{7}$$

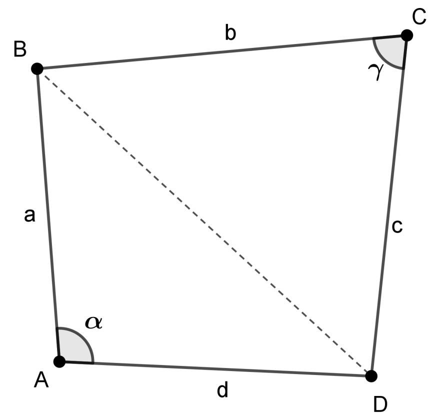


Figure 3: A general convex quadrilateral ABCD.

$$
\begin{aligned}
&= \frac{a^2+d^2+2ad-b^2-c^2+2bc(1-2\sin^2\frac{\gamma}{2})}{4ad} \\
&= \frac{(a+d)^2-(b-c)^2-4bc\sin^2\frac{\gamma}{2}}{4ad} \\
&= \frac{(a+d+b-c)(a+d-b+c)-4bc\sin^2\frac{\gamma}{2}}{4ad} \\
&= \frac{1}{ad}\left(\frac{a+b+c+d}{2}-c\right)\left(\frac{a+b+c+d}{2}-b\right)-\frac{bc\sin^2\frac{\gamma}{2}}{ad} \\
&= \frac{(s-b)(s-c)-bc\sin^2\frac{\gamma}{2}}{ad}.
\end{aligned}
$$

□

As in Theorem 1, the other formula can be obtained similarly by replacing $\cos^2\frac{\alpha}{2}$ by $1-\sin^2\frac{\alpha}{2}$ in (7).

In the case of a cyclic convex quadrilateral, you get (1) by replacing $\frac{\gamma}{2}$ by $90^\circ-\frac{\alpha}{2}$ in (5) and (6), since $\alpha+\gamma=180^\circ$.

**Theorem 6** (Bretschneider)**.** *Given a general quadrilateral with sides $a$, $b$, $c$ and $d$. If $\alpha$ and $\gamma$ are two opposite angles, then the area is given by the formula*

$$\Delta_3 = \sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\left(\frac{\alpha+\gamma}{2}\right)}.$$

*Proof.* Multiplying (5) and (6) we get

$$\left(ad\sin^2\frac{\alpha}{2}+bc\cos^2\frac{\gamma}{2}\right)\left(bc\sin^2\frac{\gamma}{2}+ad\cos^2\frac{\alpha}{2}\right)=(s-a)(s-b)(s-c)(s-d).$$

Expanding, factorizing, completing the squares and keeping in mind some well-known trigonometric identities,

$$abcd\cos^2\left(\frac{\alpha+\gamma}{2}\right)+\left(\frac{ad\sin\alpha}{2}+\frac{bc\sin\gamma}{2}\right)^2=(s-a)(s-b)(s-c)(s-d).$$

Since the area of $ABCD$ can be expressed as the sum of the areas of $\triangle ABD$ and $\triangle CBD$, which in turn can be written as $\frac{ad\sin\alpha}{2}+\frac{bc\sin\gamma}{2}$, then we are done. $\square$

It is interesting to note the resemblance of these area theorems to the identities (4), (1), (5) and (6). Indeed, as Heron's formula and Brahmagupta's formula are both special cases of Bretschneider's formula, in the same way, the identities (4) and (1) are both special cases of the identities (5) and (6). Actually, this better explains the development Heron-Brahmagupta-Bretschneider. Finally we wonder how many other interesting implications the identities (5) and (6) could have. What other research project could they inspire? For example, shall it be possible to obtain analogous identities in spherical or hyperbolic geometry?[1]If so, how would they relate to other well-known identities in such geometries? We leave the reader with these intriguing questions in mind.

# References


[1] R. C. Alperin, Heron's Area Formula, *The College Mathematics Journal* **18(2)** 137–138.

[2] G.A. Bajgonakova and A. Mednykh, On Bretschneider's formula for a spherical quadrilateral, *Matematicheskie Zametki YAGU* **1** (2012).

[3] G.A. Bajgonakova and A.Mednykh, On Bretschneider's formula for a hyperbolic quadrilateral, *Matematicheskie Zametki YAGU* **1** (2012).

[4] J. P. Ballantine, Note on Hero's formula, *Amer. Math. Monthly* **61** (1945) 337.

[5] C. A. Bretschneider, Untersuchung der trigonometrischen Relationen des geradlinigen Viereckes, *Archiv der Mathl* **2** (1842) 225–261.

[6] J. Casey, A Treatise On Plane Trigonometry, 1888, pp. 185-186.

[7] J. H. Conway and Peter Doyle, Personal email communication, 15 December 1997, https://math.dartmouth.edu/ doyle/docs/heron/heron.txt

[8] J. L. Coolidge, A Historically Interesting Formula for the Area of a Quadrilateral, *Amer. Math. Monthly* **46** (1939) 345–347.


[1] As suggested by work by G.A. Bajgonakova and A. Mednykh [2, 3].

[9] E. J. García, Proofs and applications of two well-known formulae involving sine, cosine and the semiperimeter of a triangle, from GeoDom, https://geometriadominicana.blogspot.com/2020/06/another-proof-for-two-well-known.html

[10] A. Hess, A highway from Heron to Brahmagupta, *Forum Geometricorum* **12** (2012) 191–192.

[11] V. F. Ivanoff, Solution to Problem E1376: Bretschneider's Formula, *Amer. Math. Monthly* **67** (1960) 291–292.

[12] M. Josefsson, Calculations Concerning the Tangent Lengths and Tangency Chords of a Tangential Quadrilate, *Forum Geometricorum* **10** (2010) 119–130.

[13] M. Josefsson, The area of a bicentric quadrilateral, *Forum Geometricorum* **11** (2011) 155–164.

[14] S. H. Kung, Another elementary proof of Heron's formula, *Math. Mag.* **65** (1992) 337–338.

[15] B. M. Oliver, Heron's remarkable triangle area formula, *Mathematics Teacher* **86** (1993) 161–163.

[16] V. Panagiotis, Exercise 400, *1000 General Trigonometry Exercises* **B**.

[17] C. H. Raifaizen, A Simpler Proof of Heron's Formula, *Math. Mag.* **44 (1)** 27–28.

[18] V. Thebault, The area of a triangle as a function of the sides, *Amer. Math. Monthly* **52** (1945) 508–509.

[19] P. Yiu, Euclidean Geometry, *Florida Atlantic Univesity Lecture Notes* (1998), available at http://www.math.fau.edu/Yiu/Geometry.html