Appendix A
Vector Calculus in Curvilinear Coordinates

A.1 Introduction

In this Appendix I sketch proofs of the three fundamental theorems of vector calculus. My aim is to convey the *essence* of the argument, not to track down every epsilon and delta. A much more elegant, modem, and unified–but necessarily also much longer–treatment will be found in M. Spivak's book, *Calculus on Manifolds* (New York: Benjamin, 1965).

For the sake of generality, I shall use arbitrary (orthogonal) curvilinear coordinates (u, v, w), developing formulas for the gradient, divergence, curl, and Laplacian in any such system. You can then specialize them to Cartesian, spherical, or cylindrical coordinates, or any other system you might wish to use. If the generality bothers you on a first reading, and you'd rather stick to Cartesian coordinates, just read (x, y, z) wherever you see (u, v, w), and make the associated simplifications as you go along.

A.2 Notation

We identify a point in space by its three *coordinates*, u, v, and w, (in the Cartesian system, (x, y, z); in the spherical system, (r, θ, ϕ); in the cylindrical system, (s, ϕ, z)). I shall assume the system is *orthogonal*, in the sense that the three *unit vectors*, $\hat{\mathbf{u}}, \hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$, pointing in the direction of the increase of the corresponding coordinates, are mutually perpendicular. Note that the unit vectors are *functions of position*, since their *directions* (except in the Cartesian case) vary from point to point. Any vector can be expressed in terms of $\hat{\mathbf{u}}, \hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$—in particular, the infinitesimal displacement vector from (u, v, w) to $(u + du, v + dv, w + dw)$ can be written

$$d\mathbf{l} = f\,du\,\hat{\mathbf{u}} + g\,dv\,\hat{\mathbf{v}} + h\,dw\,\hat{\mathbf{w}} \tag{A.1}$$

where f, g, and h are functions of position characteristic of the particular coordinate system (in Cartesian coordinates $f = g = h = 1$; in spherical coordinates $f = 1, g = r, h = r\sin\theta$; and in cylindrical coordinates $f = h = 1, g = s$). As you'll soon see, these three functions tell you everything you need to know about a coordinate system.

A.3 Gradient

If you move from point (u, v, w) to point $(u + du, v + dv, w + dw)$, a scalar function $t(u, v, w)$ changes by an amount

$$dt = \frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv + \frac{\partial t}{\partial w} dw; \tag{A.2}$$

this is a standard theorem on partial differentiation[1]. We can write it as a dot product,

$$dt = \nabla t \cdot d\mathbf{l} = (\nabla t)_u f\,du + (\nabla t)_v g\,dv + (\nabla t)_w h\,dw, \tag{A.3}$$

[1] M. Boas, *Mathematical Methods in the Physical Sciences*, 2nd ed., Chapter 4, Sect. 3 (New York: John Wiley, 1983).

provided we define

$$(\nabla_t)_u \equiv \frac{1}{f}\frac{\partial t}{\partial u}, \quad (\nabla t)_v \equiv \frac{1}{g}\frac{\partial t}{\partial v}, \quad (\nabla t)_w \equiv \frac{1}{h}\frac{\partial t}{\partial w}.$$

The **gradient** of t, then, is

$$\boxed{\nabla t \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{\mathbf{w}}} \tag{A.4}$$

If you now pick the appropriate expressions for f, g, and h from Table A.1, you can easily generate the formulas for ∇t in Cartesian, spherical, and cylindrical coordinates, as they appear in the front cover of the book.

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

Table A.1

From Eq. A.3 it follows that the *total* change in t, as you go from point **a** to point **b** (Fig. A.1), is

$$t(\mathbf{b}) - t(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} dt = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla t) \cdot d\mathbf{l}, \tag{A.5}$$

which is the **fundamental theorem for gradients** (not much to prove, really, in this case). Notice that the integral is independent of the path taken from **a** to **b**.

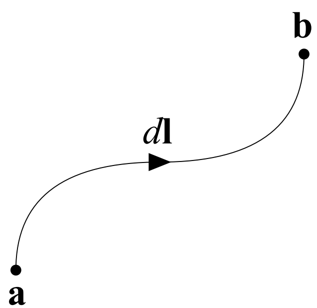

Figure A.1

A.4 Divergence

Suppose that we have a *vector* function,

$$\mathbf{A}(u, v, w) = A_u\hat{\mathbf{u}} + A_v\hat{\mathbf{v}} + A_w\hat{\mathbf{w}}$$

and we wish to evaluate the integral $\oint \mathbf{A}\cdot d\mathbf{a}$ over the surface of the infinitesimal volume generated by starting at the point (u,v,w) and increasing each of the coordinates in succession by an infinitesimal amount (Fig. A.2). Because the coordinates are orthogonal, this is (at least, in the infinitesimal limit) a rectangular solid, whose sides have lengths $dl_u = f\,du, dl_v = g\,dv$, and $dl_w = h\,dw$, and whose volume is therefore

$$d\tau = dl_u dl_v dl_w = (fgh)du dv dw. \tag{A.6}$$

(The sides are not just du, dv, dw–after all, v might be an *angle*, in which case dv doesn't even have the *dimensions* of length. The correct expressions follow from Eq. A.1.)

For the *front* surface,

$$d\mathbf{a} = -(gh)dv\,dw\,\hat{\mathbf{u}}$$

so that

$$\mathbf{A}\cdot d\mathbf{a} = -(ghA_u)dv\,dw.$$

The *back* surface is identical (except for the sign), *only this time the quantity ghA_u is to be evaluated at $(u+du)$, instead of u.* Since for any (differentiable) function $F(u)$,

$$F(u+du) - F(u) = \frac{dF}{du}du,$$

(in the limit), the front and back together amount to a contribution

$$\left[\frac{\partial}{\partial u}(ghA_u)\right] du\,dv\,dw = \frac{1}{fgh}\frac{\partial}{\partial u}(ghA_u)d\tau.$$

By the same token, the right and left sides yield

$$\frac{1}{fgh}\frac{\partial}{\partial v}(fhA_v)d\tau,$$

and the top and bottom give

$$\frac{1}{fgh}\frac{\partial}{\partial w}(fgA_w)d\tau.$$

All told, then,

$$\oint \mathbf{A}\cdot d\mathbf{a} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right] d\tau \tag{A.7}$$

The coefficient of $d\tau$ serves to define the **divergence** of $\mathbf{A}$ in curvilinear coordinates:

$$\boxed{\nabla\cdot A \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right],} \tag{A.8}$$

and Eq. A.7 becomes

$$\oint \mathbf{A}\cdot d\mathbf{a} = (\nabla\cdot\mathbf{A})d\tau. \tag{A.9}$$

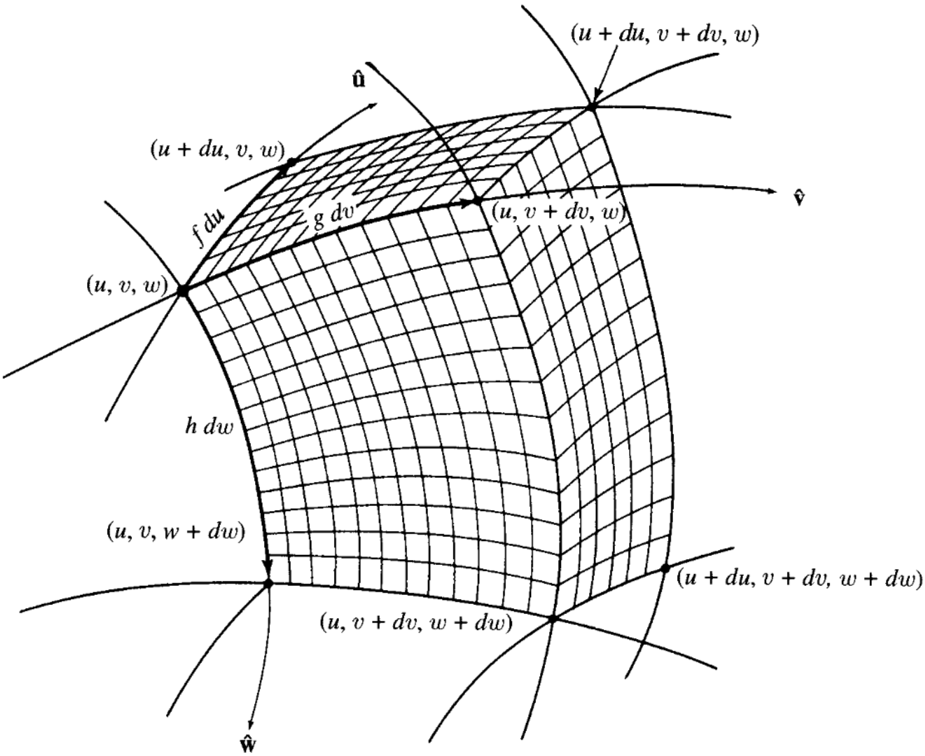

Figure A.2

Using Table A.1, you can now derive the formulas for the divergence in Cartesian, spherical, and cylindrical coordinates, which appear in the front cover of the book.

As it stands, Eq. A.9 does not prove the divergence theorem, for it pertains only to *infinitesimal* volumes, and rather special infinitesimal volumes at that. Of course, a finite volume can be broken up into infinitesimal pieces, and Eq. A.9 can be applied to each one. The trouble is, when you then add up all the bits, the left-hand side is not just an integral over the *outer* surface, but over all those tiny *internal* surfaces as well. Luckily, however, these contributions cancel in pairs, for each internal surface occurs as the boundary of two adjacent infinitesimal volumes, and since $d\mathbf{a}$ always points *outward*, $\mathbf{A} \cdot d\mathbf{a}$ has the opposite sign for the two members of each pair (Fig. A.3). Only those surfaces that bound a *single* chunk—which is to say, only those at the outer boundary—survive when everything is added up. For *finite* regions, then,

$$\oint \mathbf{A} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{A}) d\tau, \tag{A.10}$$

and you need only integrate over the external surface.[2] This establishes the divergence theorem.

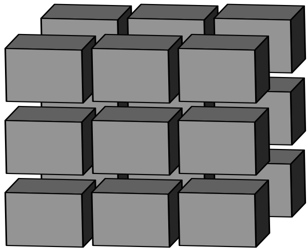

Figure A.3

A.5 Curl

To obtain the curl in curvilinear coordinates, we calculate the line integral,

$$\oint \mathbf{A} \cdot d\mathbf{l}$$

around the infinitesimal loop generated by starting at (u, v, w) and successively increasing u and v by infinitesimal amounts, holding w constant (Fig. A.4). The surface is a rectangle (at least, in the infinitesimal limit), of length $dl_u = f\ du$, width $dl_v = g\ dv$, and area

$$d\mathbf{a} = (fg) du\ dv\ \hat{\mathbf{w}} \tag{A.11}$$

Assuming the coordinate system is right-handed, $\hat{\mathbf{w}}$ points out of the page in Fig. A.4. Having chosen this as the positive direction for $d\mathbf{a}$, we are obliged by the right-hand rule to run the line integral counterclockwise, as shown. Along the bottom segment,

$$d\mathbf{l} = f du \hat{\mathbf{u}},$$

so

$$\mathbf{A} \cdot d\mathbf{l} = (fA_u)\, du.$$

Along the top leg, the sign is reversed, and fA_u is evaluated at $(v + dv)$ rather than v. Taken together, these two edges give

$$\left[-\left(fA_u\right)\big|_{v+dv} + \left(fA_u\right)\big|_v\right] du = -\left[\frac{\partial}{\partial v}\left(fA_u\right)\right] dudv.$$

[2]What about regions that cannot be fit perfectly by rectangular solids no matter *how* tiny they are—such as planes cut at an angle to the coordinate lines? It's not hard to dispose of this case; try thinking it out for yourself, or look at H. M. Schey's *Div, Grad, Curl and All That* (New York: W. W. Norton, 1973), starting with Prob. II-15.

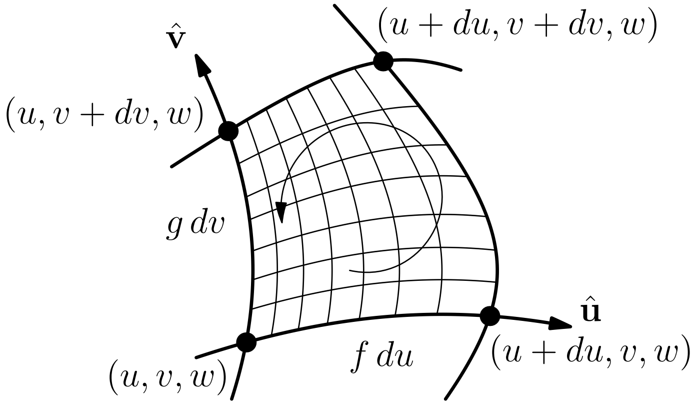

Figure A.4

Similarly, the right and left sides yield

$$\left[\frac{\partial}{\partial u}(gA_v)\right] du dv,$$

so the total is

$$\begin{aligned}\oint \mathbf{A}\cdot d\mathbf{l} &= \left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right] du dv \\ &= \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right] \hat{\mathbf{w}}\cdot d\mathbf{a}\end{aligned} \tag{A.12}$$

The coefficient of $d\mathbf{a}$ on the right serves to define the w-component of the **curl**. Constructing the u and v components in the same way, we have

$$\boxed{\begin{aligned}\nabla\times\mathbf{A} \equiv & \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} \\ & + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}\end{aligned}} \tag{A.13}$$

and Eq. A.11 generalizes to

$$\oint \mathbf{A}\cdot d\mathbf{l} = (\nabla\times\mathbf{A})\cdot d\mathbf{a} \tag{A.14}$$

Using Table A.1, you can now derive the formulas for the curl in Cartesian, spherical, and cylindrical coordinates.

Equation A.14 does not by itself prove Stokes' theorem, however, because at this point it pertains only to very special infinitesimal surfaces. Again, we can chop any *finite* surface into infinitesimal pieces and apply Eq. A.14 to each one (Fig. A.5). When we add them up, though, we obtain (on the left) not only a line integral around the outer boundary, but a lot of tiny line integrals around the internal loops as well. Fortunately, as before, the internal contributions cancel in pairs, because every internal line is the edge of two adjacent loops running in opposite directions. Consequently, Eq. A.14 can be extended to finite surfaces,

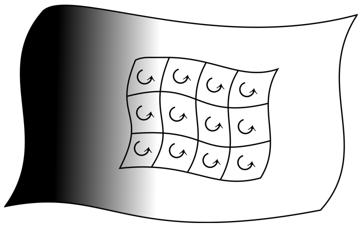

Figure A.5

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} \tag{A.15}$$

and the line integral is to be taken over the external boundary only.[3] This establishes **Stokes' theorem**.

A.6 Laplacian

Since the Laplacian of a scalar is by definition the divergence of the gradient, we can read off from Eqs. A.4 and A.8 the general formula

$$\boxed{\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right]} \tag{A.16}$$

Once again, you are invited to use Table A.1 to derive the Laplacian in Cartesian, spherical, and cylindrical coordinates, and thus to confirm the formulas inside the front cover.

[3] What about surfaces that cannot be fit perfectly by tiny rectangles, no matter how small they are (such as triangles) or surfaces that do not correspond to holding one coordinate fixed? If such cases trouble you, and you cannot resolve them for yourself, look at H. M. Schey's *Div, Grad, Curl, and All That*, Prob. 111-2 (New York: W. W. Norton, 1973).