Exploring Algebra with GrafEq

Introduction

It is our hope that those with an interest in elementary mathematics will be able to deepen and broaden their understanding of the subject through familiarity with GrafEq. The naive user of mathematical technology such as the calculator may believe that the use of such technology is limited to providing an answer. This is indeed the typical application of the scientific calculator. However, an additional application is that of *verifying* a result arrived at by standard analytic methods. The technology then serves the role of an *independent mechanical* checker. In order to satisfy this role, it is necessary that the user have some faith in the validity of the program. The software should not suffer from arbitrary limitations, nor should it provide inaccurate results.

An example is in order: suppose we hypothesize that a point has an equation and it is simply the equation of a circle with radius 0. In order to verify this via the software, it is necessary that the software be capable of plotting a single point. GrafEq can plot single points - and would indeed support this hypothesis.

A further example: if a student were to surmise that the graph of $x=|x|$ consists of quadrants I and IV over the x-y plane, (s)he could affirm this with GrafEq. This is possible only because GrafEq can plot relations of a *single* variable - a capability not typical of computer graphing software.

The user should also be aware of the inherent limitations of any physical representation of a mathematical entity. The computer screen provides a model of a region of the geometric plane, whereby *dots* in the form of pixels serve to represent geometric *points*. Pixels, like dots, have area - and therefore contain many points. An implication of this property: the graph of a complete line will appear identical to the graph of that line with a single point deleted, sometimes called a 'point discontinuity'.

Most of the concepts discussed in this booklet can be included within the branch of algebra called *analytic geometry*, wherein geometric constructs are described algebraically by equations and inequalities. Many of the topics are standard fare of a high school algebra class. However, some of the topics are *enrichment.* Whether or not the reader is a student, (s)he will benefit from considering atypical application of graphing software.

Pedagoguery Software welcomes comments and questions about both GrafEq and mathematics and invites you to contact us by e-mail at peda@peda.com

Graphics that appear unclear on screen will usually be accurate when printed.

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I Points, Lines, Segments and Rays

The basic building block of traditional geometry is the *point.* Many mathematics texts do not formally define the point. Instead, they specify properties of points. Such as: *between any two distinct points, there is a third point,* or *associated with any pair of points, there is a non-negative number*, called the distance between the points.

The *Cartesian coordinate system* allows for a convenient bridge connecting geometry and algebra. Such a coordinate system consists of two real number lines intersecting perpendicularly at their zero coordinates. The intersection is called the *origin.* These lines, called *axes,* are aligned so that the horizontal (x-) axis increases left to right and the vertical (y-) axis increases from bottom to top. The four regions of this partitioned plane are called *quadrants*, and are numbered I through IV counter-clock wise, where quadrant I is 'top-right'.

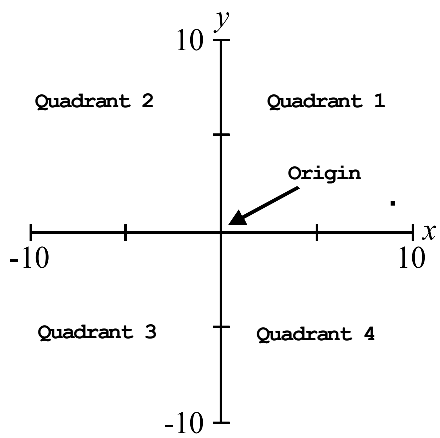

fig. 1 Cartesian coordinate system

The major advantage of this coordinate system lies in our ability to specify a point's location by two *coordinates:* the x-coordinate is the point's horizontal position with respect to the origin, and the y-coordinate is the point's vertical position. E.g. a point with x-coordinate 3 and y-coordinate -5 would be in quadrant IV, directly below 3 on the x-axis and beside -5 on the y-axis. The coordinates would be specified as the *ordered-pair* (3,-5), since the x-coordinate is written first and is followed by the y-coordinate.

Now - the question we ask ourselves: "Is there an equation whose graph consists of the single point (3,-5)?" We can, obviously, use GrafEq's constraint capability and enter the two constraint relation: x=3;y=-5. Which is equivalent to saying "x=3 *and* y=-5". But, is there a *single* equation whose solution is (3,-5)?

If we recall the definition of absolute value, then we might consider: $|x-3|+|y+5| = 0$. For this equation to hold, $|x-3|$ and $|y+5|$ must be *opposites*. But since neither can be negative, each must be zero. Which implies x is 3 and y is -5. GrafEq will confirm that we have indeed found an equation of P(3,-5). The absolute value method is not the only way to create an equation of a point. An alternative method consists of creating a circle of radius zero. Convince yourself, with GrafEq, that $(x-3)^2 + (y+5)^2=0$ is an alternative equation of the point. Are there other equations whose solution consists of (3,-5)?

Now, the equation of a line is much simpler, since it is included in the course work. Most students are aware of *slope-intercept* form: y=mx + b, where *m* is the slope of the line, and *b* is its y-intercept. The graph of y = 2x - 5 consists of the line with slope 2 which intersects the y-axis at (0,-5). Are their any lines of the x-y plane which cannot be described using slope-intercept?

It is not necessary to use this form exclusively. Consider the line passing through (4,5) and (6,3). The following

$$\frac{y-5}{x-4}=\frac{y-3}{x-6}$$

when entered into GrafEq's relation window appears to provide the desired line. However, if we examine the equation carefully, we note that x cannot be 4 or 6.[1] The correct graph is the line *minus* two points. The above equation is an example of the *2-point form* of a linear equation, whereby the line through (a,b) and (c,d) can be expressed as:

$$\frac{y-b}{x-a}=\frac{y-d}{x-c}$$

A third form, the *point-slope form* is used to create the equation of the line with slope m, that passes through (a,b) : $y-b = m(x - a)$.

The above forms have some deficiencies: either a problem with vertical lines (which have no slope) or they do not define the *whole* line.

A fourth form, sometimes called *general form* : Ax + By = C has the advantage that it can properly define any line, but this form has no simply obvious geometric properties.

We have looked at examples of lines determined by points that *belong* to the desired line. Now, as an exercise: can you determine a general equation of the *perpendicular bisector* of the segment connecting (a,b) and (c,d). (hint: the slopes of non-vertical perpendicular lines multiply to -1). Use GrafEq to test your solution by using the points (-1,2) and (5,4). Your solution line should overlap the line: y = -3x + 9.

Now, what about segments? Since segments are parts of lines, perhaps we can determine a segment's equation from the equation of its line. Suppose we wish to determine an equation of the segment PQ, where P(0,1) is one end, and Q(4,9) is the other end. An equation of the *line* through the points is y = 2x + 1. Now, if we were to

[1] Easily resolved by using the nearly equivalent cross-product: (y-5)(x-6)=(y-3)(x-4).

multiply one term by

$$\frac{\sqrt{4-x}}{\sqrt{4-x}}$$

What would be the effect?

As long as x < 4 there will be no problem. We now have a ray from Q through P. We can continue by similarly multiplying by

$$\frac{\sqrt{x}}{\sqrt{x}}$$

which similarly restricts x to be positive.

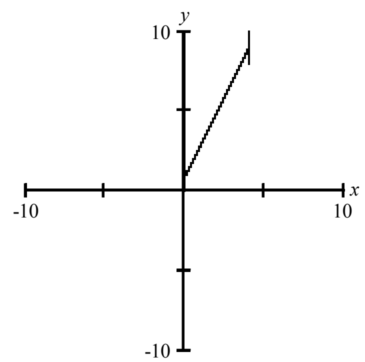

fig.2 graph of $y=2\cdot x+1\cdot\left(\frac{\sqrt{x}\cdot\sqrt{4-x}}{\sqrt{x}\cdot\sqrt{4-x}}\right)$

Note the extraneous vertical material at the end point Q(4,9). GrafEq has not yet determined that points near the end are - or are not solution points. If you select the *Information view tool - relation 1* and turn on *Show work* it becomes obvious that the program is not yet decided on the status of these points.[2]

The foregoing method is based on the simple concept of *domain restriction*. We have multiplied by a fraction that is equal to one - provided x is between 0 and 4. Indeed, unless x is between these values, the whole expression is not valid.

Is this the *only* way to arrive at the equation of a segment? Let us examine the notion of *between-ness*: The points of segment PQ are those points p *between* P and Q. And this can be expressed algebraically by saying the distance from P to p plus the distance from p to Q add to the total distance from P to Q. *The total equals the sum of the parts.* Now, we can use the distance formula for points on the x-y coordinate system:

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[2]Note that simply graphing the segment can be done easily by GrafEq's conditional capability: y={2x+1 if 0<x<4}

The distance between (a,b) and (c,d) = $\sqrt{(d-b)^2+(c-a)^2}$

So, if we put our point (x,y) between P and Q, our equation will be:

$$\sqrt{x^2+(y-1)^2}+\sqrt{(x-4)^2+(y-9)^2}=4\cdot\sqrt{5}$$

where $4\sqrt{5}$ is the distance from P to Q.
Note that the graphing side-effects of this method are different from those of the preceding method.

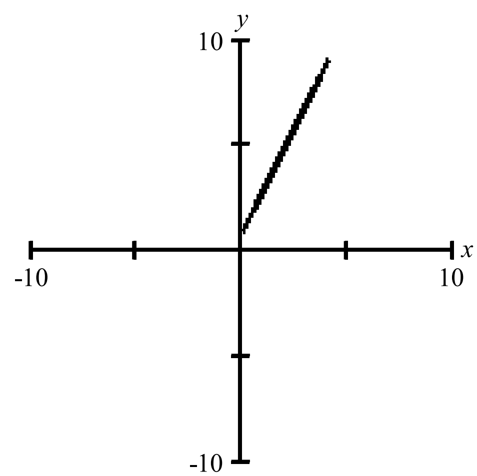


fig. 3 Graph of segment defined via distance formula

II Angles, Triangles, Unions and Intersections

We have seen how to derive equations of points, rays, segments and lines. We note that an angle is the *union* of two rays and a triangle is the *union* of three segments. How can we derive an equation for a *union*?

Let us first consider a slightly simpler case: the 'X' formed by the union of the lines: y = x and y = -x + 1.

Our equation: (y - x)(y + x - 1) = 0 See fig.4

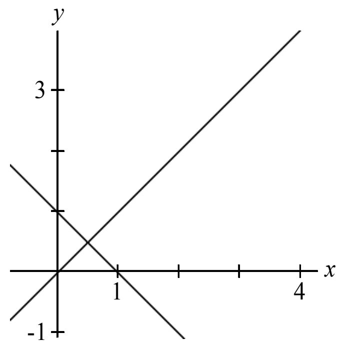


fig.4 (y - x)(y + x - 1) = 0

This equation was derived by using the well-known property a=0 or b=0 iff ab=0, where the 'iff', pronounced "if and only if" means simply that each side implies the other. Now, if we re-write y = x to y - x = 0 and y = -x + 1 to y + x - 1 = 0 we have expressions in the form 'f = 0'. These expressions can only be true if (and only if) the product of the left sides is also zero. We can summarize:

The union of the relations f = 0 and g = 0 is the relation fg = 0.

If we reflect back, on our technique for determining the equation of a point, we will realize that we *intersected* two lines to determine a point. We can similarly summarize:

The intersection of f = 0 and g = 0 is the relation |f| + |g| = 0[3]

So, an equation of an angle might be:

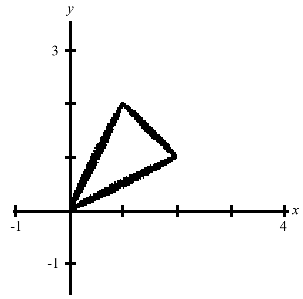
$$\frac{(y-x)\cdot(y+x-1)\cdot\sqrt{2\cdot x-1}}{\sqrt{2\cdot x-1}}=0$$

which is the union of two lines, with domain restricted, leaving an angle.[4] Now, can

[3] Or $f^2+g^2=0$, etc.

[4] More precisely, an angle minus its vertex.

we create an equation of the triangle A(0,0)B(1,2)C(2,1)? Is it not simply the union of three segments?



$$\left(\sqrt{x^2+y^2}+\sqrt{(x-1)^2+(y-2)^2}-\sqrt{5}\right)\times$$
$$\left(\sqrt{x^2+y^2}+\sqrt{(x-2)^2+(y-1)^2}-\sqrt{5}\right)\times$$
$$\left(\sqrt{(x-1)^2+(y-2)^2}+\sqrt{(x-2)^2+(y-1)^2}-\sqrt{2}\right)=0$$

fig. 5 a triangle and its equation

As an exercise, determine an equation of the square A(0,0)B(1,0)C(1,1)D(0,1) and confirm your solution by graphing your equation with GrafEq.

An aside: since we now realize that points have equations, and we can graph the *union* of any number of points, then this implies that a (monochrome) newspaper photo has a defining equation! Since, upon close examination, we see that the photo is simply a collection of dots. Needless to say, ‘zooming’ in to a graph so defined will not be the same as enlarging a photo, insofar as the graph’s ‘points’ will separate - whereas a photo’s ‘dots’ would appear larger.

III Polynomials

The following may be referred to as polynomial functions:
$y = 3x^2 - x + 1$ (degree 2)
$y = x^5 + x^3$ (degree 5)
$y = -x^7 + x^6 - x - 1$ (degree 7)
The *degree* is the highest power of x. Below in fig. 6 we see the graph of a '*cubic*'
(i.e. Degree 3)

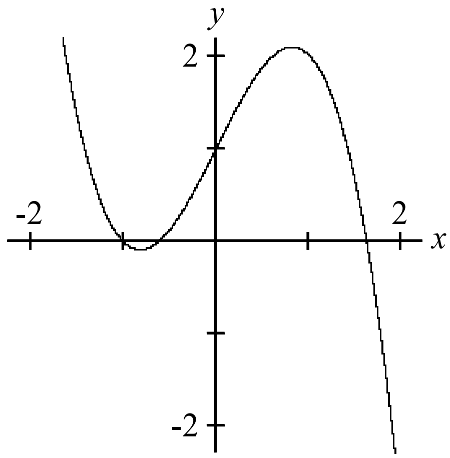


fig.6 The cubic: $y=-x^3+2x+1$

The above equation is said to determine a *function*, since, for any particular x-value, there is no more than one y-value. A 'graphical' definition of a function: "A curve is the graph of a function, *if no vertical line intersects the curve more than once*." This is often referred to as the '*vertical line test'* for functions. We are often interested in '*intercepts*': the y-intercept, which has already been mentioned, alludes to the point where a curve intersects the y-axis. Similarly, an x-intercept is determined by the intersection of the curve with the x-axis. In fig.6 above, there appears to be a single y-intercept at 1, and three x-intercepts: at -1, near -.5 and near 1.7. The x-intercepts are often referred to as 'zeros*'* since they denote x-values of those points with y-coordinates of zero.

Is there a relation between the *degree* of a function, and the number of zeroes? Use GrafEq to graph a number of functions of *odd* degree (such as $y = 2x + 1$, $y = x^5 - 3x^2 + 2$ and $y = x^7 - x^6 + 2x^3 - 6$ etc.) Do any of them appear to *not* have any zeros? Now, use GrafEq to graph some functions of *even* degree. (Such as $y=2x^2+1$, $y=-x^4 - x^3 + 2x - 3$ etc.) You should note that by modifying the constant term, it is possible to 'move' the graph so that there are no zeros at all.

In general, a polynomial equation of degree n may have *up to* n zeros and if the degree is odd, there must be *at least* one zero.

Now, there is a similar set of terms: *even or odd functions*, which should not be confused with functions of even or odd degree. An even polynomial is one in which all x-terms are of even power: e.g. $y = 2x^8 - x^6 + x^2 + 3$ and $y = 5x^{10} - 3x^6$; whereas an *odd* function is one with all x-terms of odd power: e.g. $y = 4x^7 - 2x^5 - x$ and $y = x^{11} - x^3 + x$

Fig.7 below illustrates an *even* function. We note that there is axial symmetry.

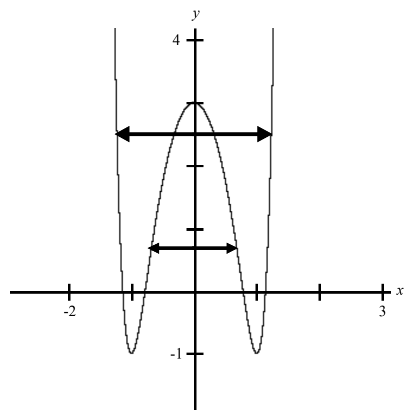


fig.7 Even function

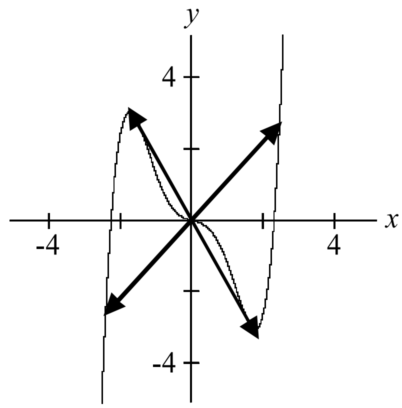


fig. 8 Odd function

The axis of symmetry of the even function is the y-axis. Every point (a,b) of the function has a symmetry point (-a,b). The origin is the point of symmetry of the odd function. Each point (a,b) of that function will have a symmetry point (-a,-b).

Do these properties generalize to non-functional relations? Take a few moments to graph 'even' relations such as $y^4-3y^3=4x^6-6x^2+1$, where we restrict the x-exponents to even integers.

Now experiment with 'odd' relations, like $y^4-3y^3=4x^5-6x^3$. Can you hypothesize about the effect of only even x-powers, only even y-powers, only odd x-powers and only odd y-powers? Can you *prove* your hypothesis?

IV Conics

These four plane curves are referred to as the *conic sections*: circle, parabola, ellipse and hyperbola. The name *conic* is due to the idea that each of these curves can be visualized as the intersection of a plane with a pair of congruent *cones* sharing a common axis and vertex.

A geometric definition: a **conic** is the *locus* of point *P* such that the ratio of the distance from *P* to a given point *Q* to the distance from *P* to a given line *l* is constant. In fig. 9, that ratio d/D (called the ***e****ccentricity) is .5*

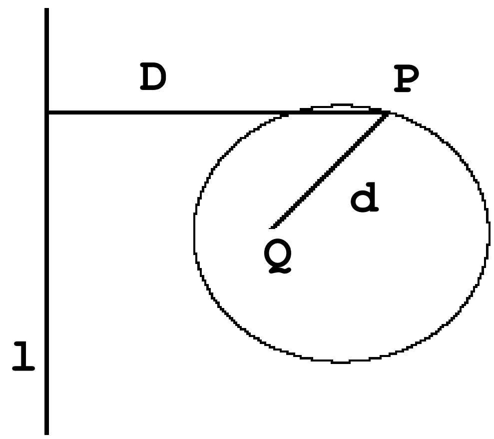


fig.9 conic with *focus Q* and *directrix l.*

The conic is an *ellipse* if 0<e<1; a *parabola* if e=1 and a *hyperbola* if e>1. We should note that this definition, based upon *e*, precludes a circle being a conic. We can nevertheless consider a circle to be the *limiting* case as D increases to infinity. I.e. as e approaches zero.

With GrafEq, use the following three constraint relation: and replace E with various positive constants in order to see the effect. Note that in this example, we are setting the focus at (1,0) and the directrix is the y-axis. We use 'E', rather than 'e', since 'e' is reserved for the exponential constant 2.7183….

$$d=\sqrt{(x-1)^2+y^2}$$

$$D=|x|$$

$$\frac{d}{D}=E$$

and replace E with various positive constants in order to see the effect. Note that in this example, we are setting the focus at (1,0) and the directrix is the y-axis.

An *algebraic definition*: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is the *general quadratic equation in x and y*, and its graph (if it exists) is a *conic* (parabola. ellipse, circle or hyperbola) or *degenerate conic* (two intersecting lines, a line or a point). Use GrafEq to convince yourself that:

for A=1, C=2, and B=D=E=F=0 the graph is a point.
for A=1, C=2, F=1, and B=D=E=0 the graph is empty.
for A=1, B=-1, C=-2, D=-1, E=11, F=-12 the graph is a pair of intersecting lines
for A=B=C=0 the graph is a line.

Can you visualize how a plane might intersect the cone-pair at a point, line or 'X'?

Note that in the general quadratic, where A and C are not both zero, the effect of the xy term includes a *rotation* of the curve.

Use GrafEq to study the graph of : $4x^2 + Bxy - y^2 + 2x - y - 5 = 0$ for various values of B, such as 0, 5 and -5.

We can experiment with the preceding concepts as follows, using the basic equation: $x^2 + y^2 = z^2$ whose graph is the 3-dimensional bi-cone. If we enter the following two-constraint relation,

$x^2+y^2=z^2$
$y=\{0,1,4\}$

and examine the x-z view we see an 'X' corresponding to the y=0 plane, and two hyperbole corresponding to the y=1 and y=4 planes.

If we then edit the relation to be:

$x^2+y^2=z^2$
$z=\{0,1,4\}$

and examine the x-y view, we see two circles corresponding to the z=1 and z=4 planes, and a single point corresponding to the z=0 plane.

If we now enter the following

$x^2+y^2=z^2$
$z=\{.1,.2,2\}\cdot y+3$

and use the x-z view, we will see two ellipses and a hyperbola. The ellipses are determined by the planes: $z = .1y + 3$ and $z = .2y + 3$. The 'steeper' plane $z = 2y + 3$ determines the hyperbola. The graph is illustrated in fig.10 below.

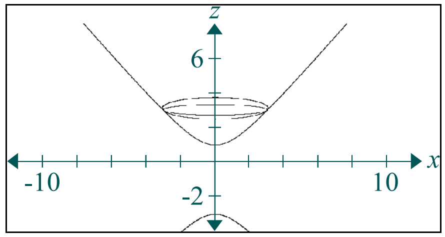


fig.10 intersection of bi-cone by oblique planes

The preceding examples have not illustrated the parabola. If we enter the following two-constraint relation:

$x^2 + y^2 = z^2$

$z = y + 3$

and view it from the three viewports: x-y, x-z and y-z we will see two parabolas and one ray. Can you explain why one view is a ray? Is there a perspective in 3-D in which the parabola in space will *appear* as a *line*? A *segment*? Note that the plane: z=y+3 is 'parallel' to the edge of the bi-cone and therefore only intersects one of the cones. If we now 'solve' the 2-constraint relation, we derive: $y=1/6\ x^2 - 1.5$ whose graph matches the original parabola as seen in the x-y view.

V Solving equations

The technique used by most graphing technology to solve equations in one variable , such as f(x)=0, is to construct a 2X2 system: y=f(x) and y=0 and proceed to determine the x-coordinates of the intersection. We note that GrafEq is capable of displaying a one-variable relation directly on the two-variable plane. Therefore, to solve x^3-$2x^2$ = x-2 we simply enter it and graph, to display three vertical lines corresponding to x= 1, 2 and -1. Of more interest, perhaps, is the exercise of solving an equation in one variable over the *complex* numbers. For example if we wish to determine all cube roots of -1. (x^3=-1) We can proceed as follows: let a solution be: a+bi and substitute this expression for x:

$(a + bi)^3 = -1$ and expand to get

$a^3 + 3a^2bi + 3ab^2i^2 + b^3i^3 = -1$which reduces to:

$a^3 + 3a^2bi - 3ab^2 - b^3i = -1 + 0i$

Equating the real and imaginary parts on both sides of the equation:

$a^3 - 3ab^2 = -1$ and $3a^2bi - b^3i = 0i$

So, our two-constraint relation: $a^3 - 3ab^2 = -1$; $3a^2b - b^3 = 0$ may be graphed as in fig. 12 below.

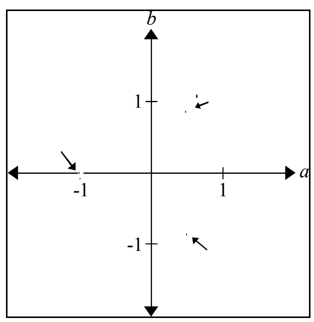


fig.12 Three complex solutions of x^3=-1 as displayed on the a-b plane

The three solutions in fig.12 above (-1,0),(.79,.79) and (.79,-.79) correspond to -1+0i and .79±.79i.

GrafEq can also be used to solve *systems of equations* with more than two variables. Consider the following system, whose solution consists of a 4-tuple:

w^2-2x+y+3z = 18

x-y-2w+z = 6

-x+y+z = w+1

x+y = 3w-9

If the user simply enters the above as a 4-constraint relation, and firstly examines the x-y view, and subsequently the w-z view via the 1-point mode (s)he will determine the solution approximating (3, -1, 1, 2)

VI Graphing Inequalities Without Inequality Signs

It is an interesting property of the absolute value function that permits the graphing of regions through equations. For instance, the graph of $y=x^2+|y-x^2|$ consists of those points on and above the parabola $y=x^2$. See fig. 13 below.

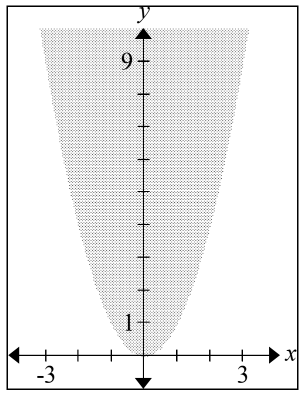


fig. 13 graph of $y=x^2+|y-x^2|$

Why does this happen? The definition of absolute value permits a number to equal its absolute value, only if it is greater or equal to zero.

$$\mathbf{f \geq 0 \quad \text{is equivalent to } f = |f|}$$

So, $y \geq x^2$ is equivalent to $y - x^2 \geq 0$ which is equivalent to $y - x^2 = | y - x^2|$

VII Locus

The path of a point that moves according to some rule is often referred to as a *locus*. For instance, the locus of a point on the x-y plane whose distance from (3,2) is exactly 5 is a circle. We are often interested in the graph of the locus and its equation.

The multi-constraint capability of GrafEq greatly simplifies the plotting of loci.

Let us consider the following example:

> Determine the graph and equation of the locus of a point on a ladder as the ladder slides smoothly down the wall from vertical to horizontal**.** The point is 1/4 the length of the ladder below the top of the ladder. The ladder is 8 m. in length**.**

Solution: Without loss of generality, we will consider the wall to be the y-axis and the floor to be the x-axis. So, the top of the ladder is originally at (0,8) and the bottom at (0,0). Let A(0,a) be the top of the ladder, B(b,0) be the bottom and P(x,y) be our locus point. Our defining relation, in three constraints:

$\sqrt{(0-a)^2+(b-0)^2}=8$ by the distance formula, since AB = 8

$\frac{y-a}{x-0}=\frac{a-0}{0-b}$ by the slope formula, since P is on line AB

$\sqrt{(x-0)^2+(y-a)^2}=2$ by the distance formula, since PA = 2.

The graph of the relation defined above:

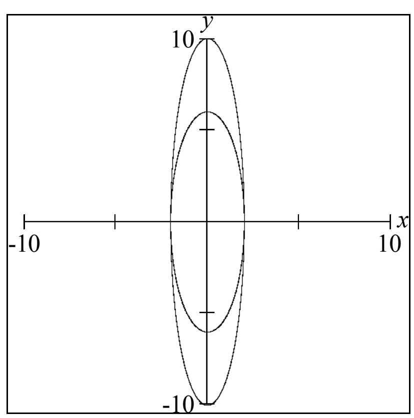


fig. 14 graph of the locus of a point on a sliding ladder.

We note that our three constraints have not restricted the locus to quadrant I. More interestingly, we appear to have two curves - because we have not restricted point P to be between A and B. We can restrict P to be between A and B by adding a fourth constraint: $\sqrt{(x-b)^2+(y-0)^2}=6$

which restricts P to be six away from B[5]. Our graph now *appears* to be an ellipse, with intercepts at (0,6), (0,-6), (2,0) and (-2,0). See fig. 15 below.

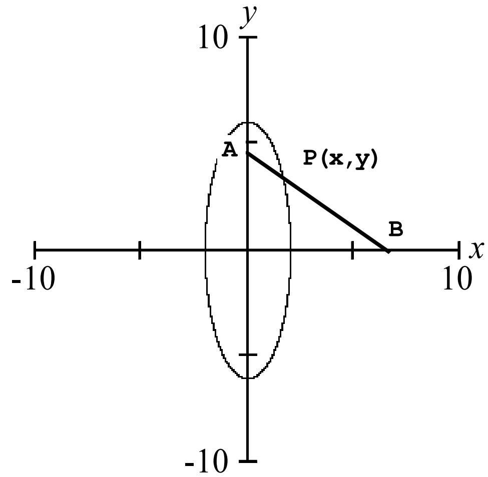


fig. 15 locus of point P on ladder AB

Now, if we wish to determine an equation of the curve, we might use algebraic techniques to *solve* the system of constraints above. However, we can possibly anticipate the equation by presuming it is indeed an ellipse passing through those four points listed above. Then, using the standard equation of an ellipse with centre (0,0) and y-semi-major axis 6 and x-semi-minor axis 2 we derive:

$\frac{(x-0)^2}{2^2}+\frac{(y-0)^2}{6^2}=1$ Standard equation of ellipse

If we superimpose the graph of the above relation on top of our original relation, we note that they seem to coincide. Indeed, if we repeatedly zoom into any part of the curve, we cannot separate the relations. Although obviously not a *formal* proof - this exercise seems most compelling that our standard equation above is indeed the equation of our locus. We do note that we have allowed our ladder top to slide below the floor and our ladder bottom to slide horizontally in either direction. A formal algebraic derivation of our standard equation above, from the original constraints is left as an exercise for the reader. Once we conclude that equation above is correct, we can confidently say that the locus is indeed an ellipse. We have 'discovered' a geometric definition of the ellipse which is quite different from the standard definition involving two foci and a sum of focal radii[6]

---

[5] The discriminating reader will notice that by adding the fourth constraint, we may now no longer need the second constraint - since constraints three and four together guarantee that point P will be on line AB. So - if we retain the last constraint and delete the second we should still achieve our desired graph. But if we do so, our graph has a *calligraphic* appearance. It is often desirable to *over-constrain* a relation when using GrafEq.

[6] A subject for further study: what, if any, is the relationship between the foci and sum of focal radii in

VIII Circular & Hyperbolic Functions Etc.

We can say that the sine function *maps* the arc length l to the y-coordinate of end-point P, on the unit circle and the cosine function similarly maps l to P's x-coordinate.

An alternative definition of these circular functions is based on *area*. By this method, we may say that the sine function maps twice the area of the sector bounded by radius OP, the x-axis, and the subtended arc to the y-coordinate of P. The cosine function similarly maps 2A to P's x-coordinate. (See fig. 17) It can be shown that these definitions are equivalent insofar as they generate the same values.

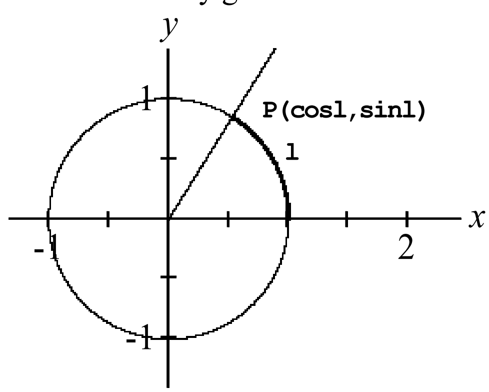


fig. 16 Circular functions derived from arc length

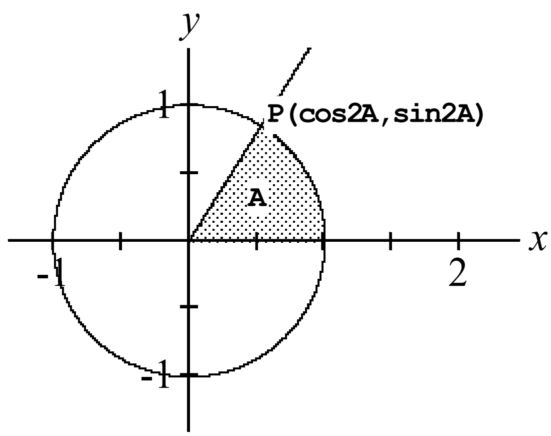


fig. 17 Circular functions derived from area

Students who notice the sinh (HYP-sin) and cosh (HYP-cos) buttons on their calculators and proceed to look up the standard definitions will see that they are based upon exponentials of the number *e*.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

---

the standard definition - and the length of segment and relative position of the locus point in the 'sliding ladder' definition?

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

Now - both of the above definitions of the circular functions are based on the unit circle ($x^2 + y^2 = 1$). We may now proceed to define the hyperbolic functions based on the hyperbola : $x^2 - y^2 = 1$. And we use the area-based method. In fig. 18 below, the cosh function maps 2A to P's x-coordinate and sinh maps 2A to the y-coordinate.

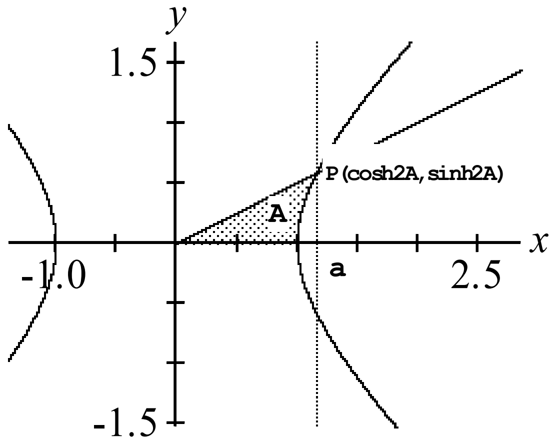


fig.18 Hyperbolic Functions derived from area

The reader can verify the above technique by using the formula for the area A below:

$$A = 1/2 \ln\left|a + \sqrt{a^2 - 1}\right|$$

and then using a calculator to derive cosh(2A) and sinh(2A)

a	Area A	cosh(2A)	sinh(2A)	$\sqrt{a^2 - 1}$
1	0	1	0	0
2	.658...	2	1.732...	1.732...
3	_______	_______	_______	________
4	_______	_______	_______	________

We note that a equals cosh2A and sinh2A is the root of $a^2 - 1$.(the height of the hyperbola at x=a.)

We can now use this same area-based definition to create *diamond* and *square* functions.

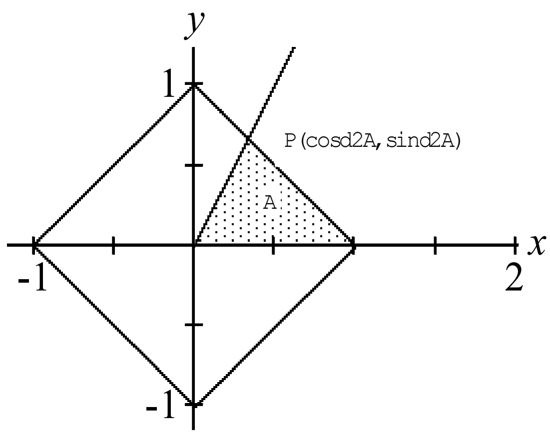


fig.19 Diamond functions defined by |x| + |y| = 1

The graphs of y = sind x and y = sinq x (*Square* function) are as shown below.

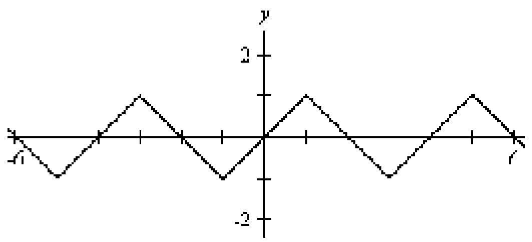


fig.20   y = sind x

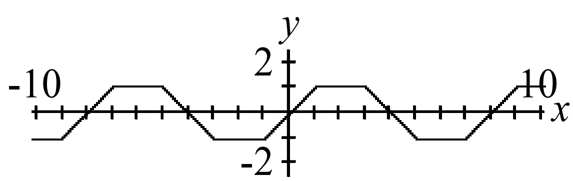


fig.21   y = sinq x

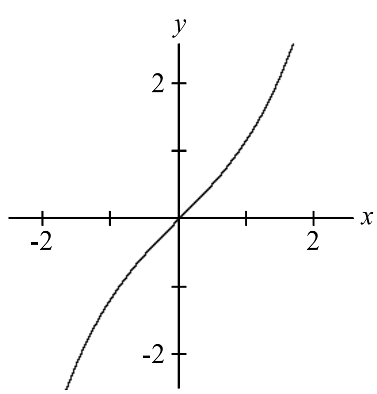


fig.22   y = sinh x

The square functions are based on the square with vertices at (1,1), (-1,1), (1,-1) and (-1,-1), using the same double area technique.

We note that sind and sinq are similar to the sin function in that they are periodic, with an amplitude of 1. The hyperbolic functions, however, are not periodic, but they do share the property that the derivative of sinh is cosh (as the derivative of sin is cos).

IX Envelopes

A mathematical *envelope* can be defined as a curve that is tangent to every member of a family of curves. For example, if we consider the set of coplanar segments of length l sharing a common end-point P and consider the perpendicular lines at the other ends of these segments, then the envelope of these perpendiculars consists of the circle with centre P and radius l.

An interesting envelope is that formed as follows: within a circle we select a point P (not the centre) and draw chords through P. At each chord's end points we construct perpendiculars. The envelope of these perpendiculars is an ellipse. (fig.23)

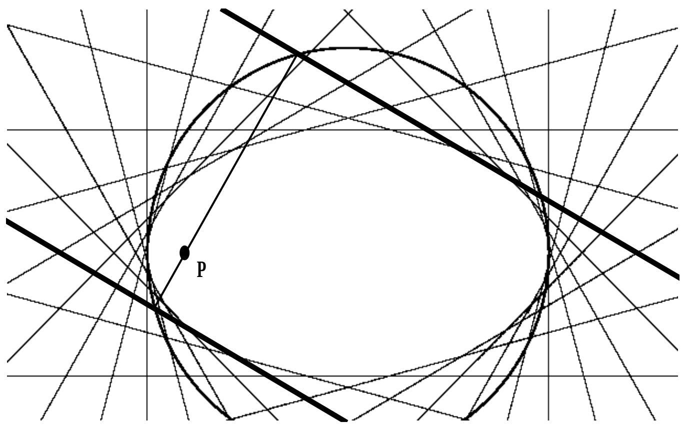


fig. 23

$$a=\frac{k\cdot\pi}{12}$$

$$\frac{b}{a+4}=\tan\alpha=\frac{a-x}{y-b}$$

$$a^2+b^2=25$$

$$k=\{1,2,3,4,5,6,7,8,9,10,11,12\}$$

The 4-constraint relation for fig. 23 defines the relation consisting of 12 pairs of perpendiculars. The circle in the relation has centre (0,0) and radius 5. P is (-4,0). The chord's end point is (a,b).The slope of the chord is tan α

Now, it certainly seems plausible that the illustrative case described above consists of the ellipse with centre (0,0), focus P(-4,0), major axis of length 10 and minor axis of length 6. We can speculate that this is the case by noting that the horizontal chord through P will intersect the circle at (-5,0) and at (5,0), and at these points the perpendicular tangents are vertical lines. Similarly, the vertical chord through P will intersect the circle at (-4,3) and at (-4,-3). The perpendiculars formed at these points will be horizontal lines passing through (0,3) and (0,-3). Now this is certainly no formal proof, but let us consider, say, the chord through P(-4,0) terminating at (0,5). Will the perpendicular formed be tangent to this ellipse? Let us consider the intersection of the perpendicular line L: (y-5) = -.8x with the ellipse: (x/5)2 + (y/3)2 = 1 If the line is tangent to the ellipse we anticipate a single solution. The most convenient technique using GrafEq is to define a 2-constraint relation

$$\left(\frac{x}{5}\right)^2+\left(\frac{y}{3}\right)^2=1$$
$$y\text{-}5=\text{-}.8\cdot x$$

and examine its graph. The graph of the above relation, after a number of 'zoom-ins' does appear to be a single point. The point's coordinates are approximately (4,1.8) - the point directly above the other focus (4,0).[7]

Assuming our conclusions are true, there is one other item of note in this topic: we now have a method for geometrically constructing a tangent to a given ellipse at any point other than the ends of the major axis.

The method is as follows: given an ellipse with centre C, foci F and F',and major axis AB we can construct a tangent through P by the following steps (note fig. 24)
1. Construct the circle with center C and radius AC
2. Construct segment PF, where P is on the ellipse.
3. Construct line QC, which bisects FP with Q on the circle
4. Line QP is tangent to the ellipse.[8]

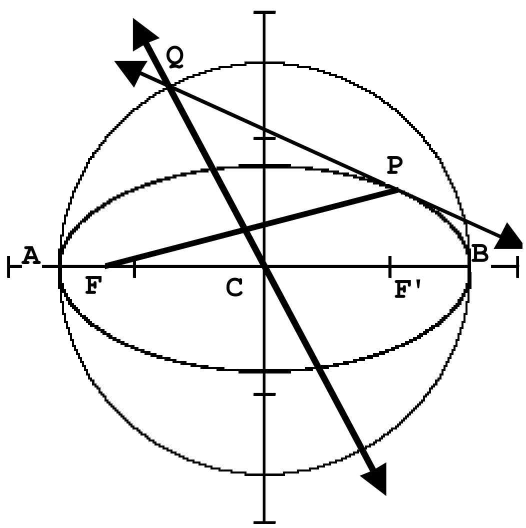


fig. 24

[7] Solving $(x/5)^2 + (y/3)^2 = 1$ and y - 5 = -.8x simultaneously yields
$x^2 - 8x + 16 = 0$ whose single solution is x = 4.

[8] Note that this is not the simplest way to construct a tangent to an ellipse. The more direct method for constructing the tangent at P is to construct the perpendicular to the angle bisector of ∠ FPF'

As a further example of an envelope, we consider the family of lines PM which perpendicularly bisect the segments connecting a given point F and a point Q on the line L. The envelope of this family is the parabola with focus F and directrix L.

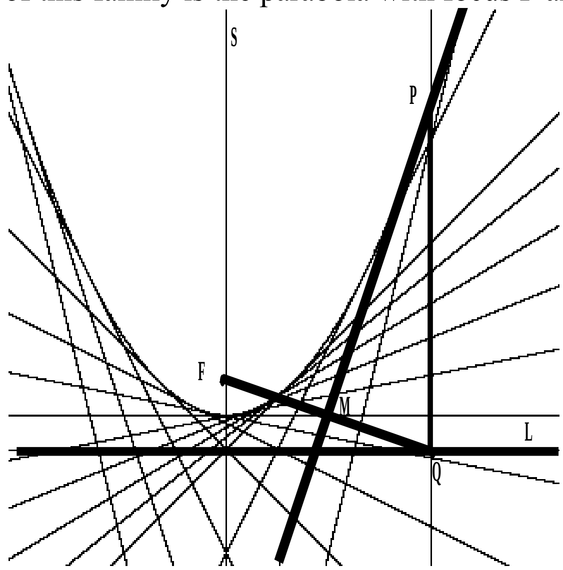


fig. 25[9]

$$y-\frac{f}{2}=\frac{a}{f}\cdot\left(x-\frac{a}{2}\right)$$
$$f=1$$
$$a=\{-4,-3,-2,-1,-.5,-.2,0,.2,.4,.6,.8,1,2,3,4\}$$

Point P, the intersection of the perpendicular to L at Q with the perpendicular bisector of FQ, is a point of the parabola. An implication of this construction is that for a 'vertical' parabola as above, the slope of the tangent at P is simply the directed distance from the axis of symmetry S to P divided by the directed distance from the directrix to the focus. Also, as was the case with the elliptical envelope, we can use the example above to derive a construction for creating a tangent line at any point on a parabola.

Our final example of an envelope hearkens back to our previous case of the ladder sliding down the wall (p17 - locus), however we are now interested in the envelope defined by the family of such ladders. (Segments of equal length, with end-points on the perpendicular axes.)

---

[9] The defining 3-constraint relation defining the family is derived as follows: a is the set of x-coordinates of points Q and P; f is the y-coordinate of the focus; f/2 is the y-coordinate of M and a/2 is its x-coordinate; a/f is the slope of line PM (since -f/a is FQ's slope). The directrix is y=0.

In this instance, since we desire only segments, it is most convenient to graph the family of segments by using four relations - one for each quadrant. In the relation expressed below for quadrant four, *a* is the y-coordinate of the segment's end on the y-axis, and *b* is the x-coordinate of the segment's end on the x-axis.

The quadrant 4 relation: a family of 11 segments defined by six constraints

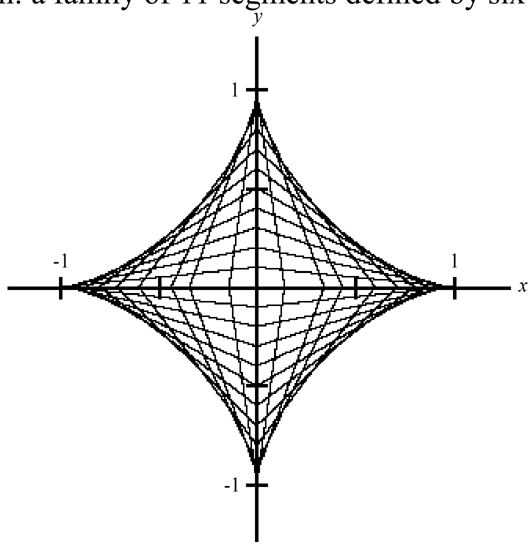


Fig. 26

$$a=\{-.99,-.94,-.9,-.8,-.7,-.6,-.5,-.4,-.3,-.2,-.1\}$$

$$a^2+b^2=1$$

$$b>0$$

$$y-a=\frac{-a}{b}\cdot x$$

$$0<x<1$$

$$-1<y<0$$

The envelope illustrated in fig. 26 is called an *astroid*.

X Coordinate Systems

The power of analytic geometry lies in the wedding of the geometric *visual* aspect of the graph of a relation with the *precision* available through algebraic abstraction. GrafEq supports two systems directly: the Cartesian rectangular and the polar.. However, we can use GrafEq to experiment with systems of our own design.

Let us consider a system of two axes (X and Y) which are not perpendicular. An X-coordinate will be the horizontal distance from the Y-axis. The Y-coordinate will be the oblique distance (parallel to the Y-axis) from the X-axis.

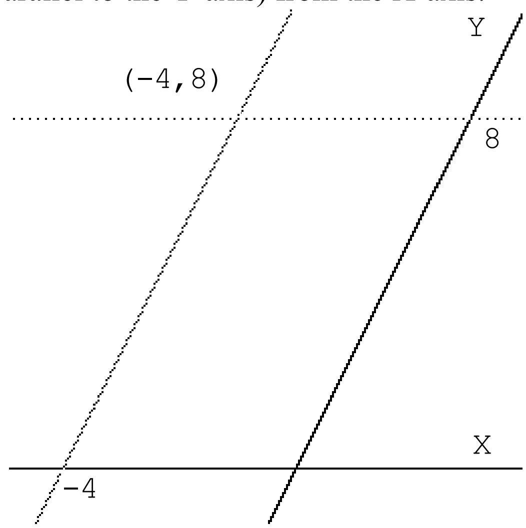


Fig. 27 Non-standard Coordinate System

Let us now graph |X| + |Y| = 8 using the coordinate system above.

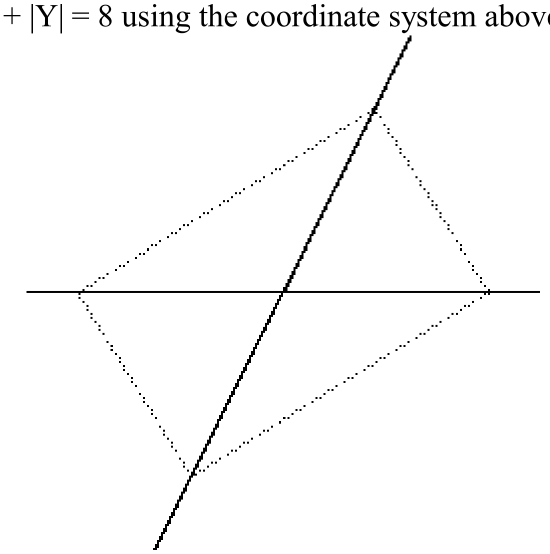


Fig. 28 |X|+|Y|=8

We note that the standard diamond (|x|+|y|=8) is transformed into a 'rectangle' whose diagonals both remain 16, as in the graph of |x|+|y|=8 on the standard Cartesian system. Let us continue to experiment further by creating the graphs of Y=1/X and $(X-5)^2+(Y-5)^2=25$:

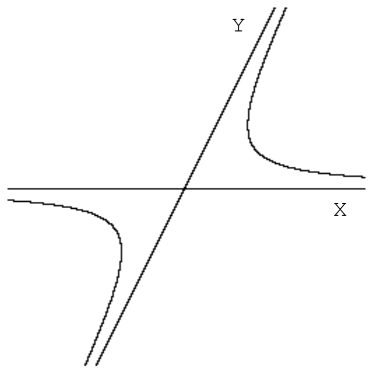


fig. 29 Y=1/X

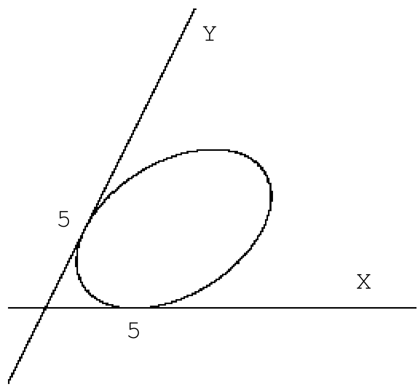


fig.30 $(X-5)^2+(Y-5)^2=25$

It appears that fig. 29 illustrates a hyperbola and fig. 30 an 'ellipse'.

How are these graphs created using GrafEq? The Y-axis has an arbitrary slope of 2 (in terms of the standard x-y system). The X-axis is the traditional x-axis. And we maintain equal scales so that (0,1) and (1,0) in the X-Y system will both appear equidistant from (0,0).

Therefore we define X and Y as follows, using two constraints:

$$x = X + \frac{Y}{\sqrt{5}}$$

$$y = \frac{2Y}{\sqrt{5}}$$

So, the X- and Y- axes may be plotted by adding a third constraint XY=0. The rectangle of fig. 28 is achieved by making |X|+|Y|=8 a constraint.

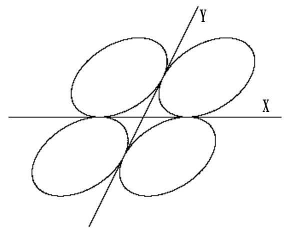


fig.31 $(X \pm 5)^2+(Y \pm 5)^2=25$

Fig.31 illustrates the effect of 'reflecting' over the axes.

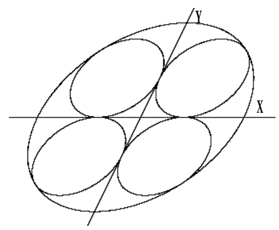


fig.32 $X^2+Y^2 = 25(1+ \sqrt{2})^2$ - an external tangent

The preceding illustrations raise a number of interesting questions:
- Is fig. 30 actually an ellipse? What are the major and minor axes lengths and how are they related to the 'radius' five?
- What is the equation of the line that bisects quadrants I and III?
- What might $Y=X^2$ or $Y=\sin X$ look like?
- What X-Y equation would produce an apparent circle?
- Can we claim that all shapes are indeed what we expect - but are *distorted* and simply appear different?

XI An Optimization Exercise

GrafEq's multi-constraint capability proves invaluable in a broad range of situations. One example is found in the problem of determining an optimal value, as in the following example:

> "Given a circular piece of paper of radius 5, a sector of central angle θ is removed, and the remaining material is transformed to a cone. What value of θ will result in the cone of maximum volume?"

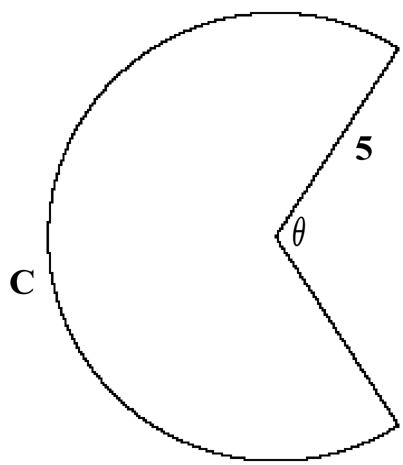


fig. 33

We can calculate the remaining circumference as follows:

$C = 10\pi(2\pi - \theta)/2\pi = 10\pi - 5\theta$ **I**

This arc of length C will be the circumference of the cone.

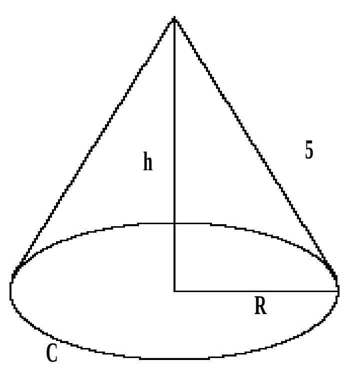


fig. 34

Therefore the radius R of the cone's base may be calculated:

$R = C/2\pi$ **II**

R may be used to calculate h:

$h = \sqrt{5^2 - R^2}$ **III**

and the volume of the cone will be:

V=1/3 π R^2h **IV**

We may now graph our relation by entering the following into a GrafEq relation window:

$$C=10\cdot\pi-5\cdot\theta$$
$$R=\frac{C}{2\cdot\pi}$$
$$h=\sqrt{25-R^2}$$
$$V=\frac{1}{3}\cdot\pi\cdot R^2\cdot h$$
$$0<\theta<2\cdot\pi$$

and plot using the view: θ vs V in **Cartesian** coordinates. We proceed to zoom out until we see the function's maximum, then zoom in and use the 1-point view tool to approximate the maximum at (1.15,50.38). Our conclusion: the cone of maximum value will be achieved by selecting θ to be approximately 1.15 radians.

We now corroborate our result using differential calculus. Rather than massage our equations I - IV into a single expression V=f(θ), which is somewhat unwieldy, we use the chain rule:

$$\frac{dV}{d\theta} = \frac{dV}{dR}\cdot\frac{dR}{dC}\cdot\frac{dC}{d\theta}$$

From IV: V=1/3 π R^2h, it follows that

$$\frac{dV}{dR} = \frac{\pi}{3}\left[R^2\cdot\frac{dh}{dR} + h\cdot 2R\right] = \frac{\pi}{3}\left[R^2\cdot\frac{-R}{\sqrt{25-R^2}} + \sqrt{25-R^2}\cdot 2R\right]$$

and from II: R = C/2π it follows that

$$\frac{dR}{dC} = \frac{1}{2\pi}$$

and from I: C = 10π - 5 θ it follows that

$$\frac{d\cdot C}{d\cdot\theta}=-5$$

Therefore,

$$\frac{d\cdot V}{d\cdot\theta}=\frac{\pi}{3}\left[\frac{-R^{3}}{\sqrt{25-R^{2}}}+2\cdot R\cdot\sqrt{25-R^{2}}\right]\left[\frac{-5}{2\cdot\pi}\right]$$

The maximum point will occur when

$$\frac{dV}{d\theta}=0$$

Which occurs if

$$\frac{R^3}{\sqrt{25-R^2}}=2R\sqrt{25-R^2}$$

This[10] reduces to $R=5\sqrt{2/3}$, which, using C=2πR and θ=(C-10π)/-5 from I, yields θ=2π(1-1/3 $\sqrt{6}$) which is approximately 1.1529… which corresponds closely to our answer determined graphically. It is worth noting in passing, that although we calculated the derivative with respect to θ, our solution was achieved in terms of R.

---

[10] Alternatively, one can create an additional 6th constraint:

$$y=\frac{R^3}{\sqrt{25-R^2}}-2R\sqrt{25-R^2}$$

and plot y vs. θ . This has the advantage that we seek the intersection of y and θ which is much easier to precisely determine, compared to finding a maximum point.

And yet another alternative: as a 6th constraint, enter

$$\frac{R^3}{\sqrt{25-R^2}}=2R\sqrt{25-R^2}$$

then graph V vs. θ, looking for a single point.

XII Transformations – An Exploration

In our curriculum we examine various transformations: translations, reflections and scalings. We here address particularly the relationship between the *algebraic* (symbolic) and the *geometric* (visual) representations of 2-D relations. The student will have learned that replacing x with a function of x results in a horizontal effect and replacing y with a function of y results in a vertical effect. The summative skill is exemplified when a student can describe the graph of, for example, $y=3\sin4(x+\pi) - 2$ in terms of the graph of $y=\sin x$: “period quartered; amplitude tripled; translated down 2 and π to the left.” The student realizes that every term in the algebraic representation has a corresponding attribute in the graphical representation.

The student may have been perplexed upon noting that such replacements seem to have the opposite effect from what might be expected: replacing x with x**-**3 results in a translation in the **positive** direction; replacing y with y/2 results in y-coordinates being **multiplied** by 2.

The question that naturally arises from the study of this topic: “Can we anticipate the effect on a given graph if we replace a variable in the original equation with an arbitrary function of that variable?” For example: What is the effect if we replace x with x^2 or with $\sqrt[3]{x}$ or with sinx?

Our method of addressing this question is essentially ‘scientific/experimental’ – we will experiment by making various replacements and note the effects. Although the graphing could be done manually we can greatly expedite matters by using GrafEq to produce all graphs.

Experiment 1:

Initial relation A: $y=\sqrt[3]{x}$

Replace x with |x| A': $y=\sqrt[3]{|x|}$

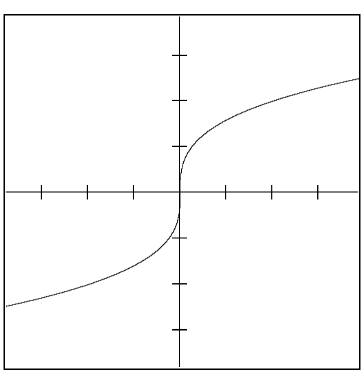


fig.35 $y=\sqrt[3]{x}$

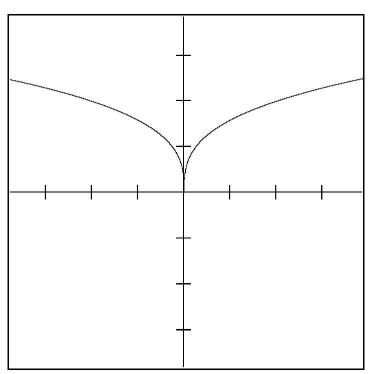


fig.36 $y=\sqrt[3]{|x|}$

It appears that the quadrant 3 portion of A has reflected over the x-axis! But this contradicts our assumption that x-replacements imply horizontal effects. Let's try the same replacement on B: $y=x^3$

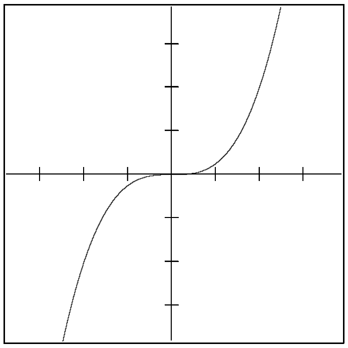


fig.37 $y=x^3$

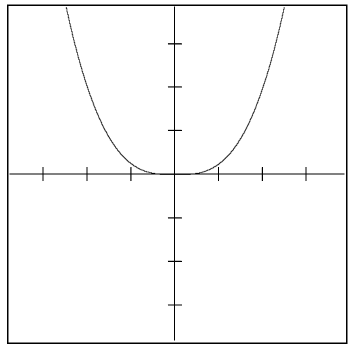


fig.38 $y=|x|^3$

This example seems to confirm the same aberration. Maybe our problem stems from the fact that both original graphs share the property that the quadrant 1 and quadrant 3 portions are point symmetric. Let us examine an example like C: $y= 2^x-1$.

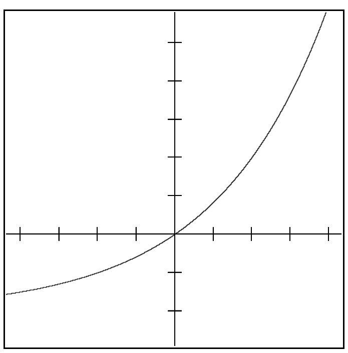


fig.39 $y= 2^x-1$

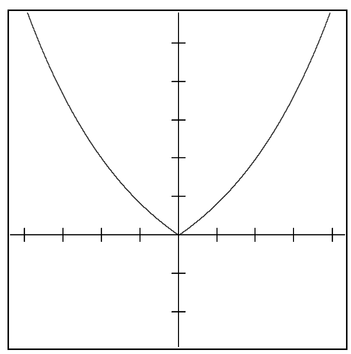


fig.40 $y= 2^{|x|} -1$

It appears that the quadrant 3 portion of C has been 'discarded' and the quadrant 1 portion has been both left alone and reflected over the y-axis. Let us next consider D: $y=x^3-3x+1.5$, which has the property that none of the quadrantal portions are similar.

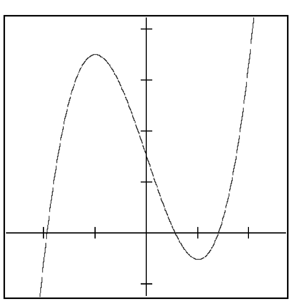


fig.41 $y=x^3-3x+1.5$

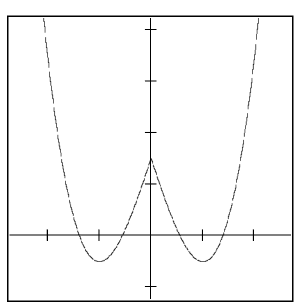


fig.42 $y=|x|^3-3|x|+1.5$

Our observation can be expanded to the whole plane: the quadrants 1 and 4 portions have remained and been reflected over the y-axis. The quadrant 2 and 3 portions are lost. Or, put another way, if (a,b) is a point on y=f(x) then if a is positive there will be two corresponding points on y=f(|x|) : (a,b) and (-a,b). But if a is negative, there are no corresponding points on f(|x|). Can we anticipate what might be the effect if we "absolute" both x and y variables? Using the basic relation in fig.41 above we can examine the result of the two replacements. The relation of fig.41 is particularly useful because each quadrantal portion is distinct from the others.

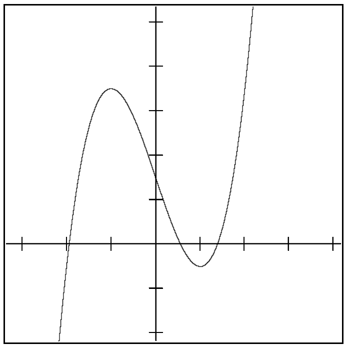


fig.43 $y=x^3-3x+1.5$

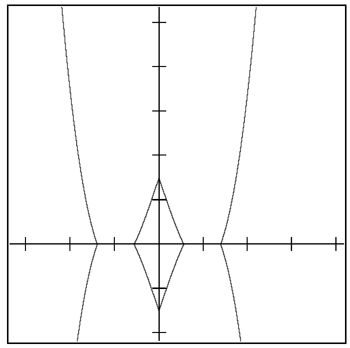


fig.44 $|y|=|x|^3-3|x|+1.5$

It appears that the graph of |y|=f(|x|) can be determined by simply examining the quadrant 1 portion of y=f(x): that portion is replicated by reflection over both axes. And the Q3 portion is simply a subsequent reflection. For (a,b) in Q1, there will be 3 additional points generated: (-a,b), (-a,-b) and (a,-b). Points in the other quadrants will be discarded. When will (c,d) be a point of |y|=f(|x|)? If and only if (|c|,|d|) is a point of y=f(x).

Now, let us examine the effect of replacing x with x^2

Experiment 2

Initial relation	D: $y=x^3-3x+1.5$
Replace x with x^2	D': $y=(x^2)^3-3x^2+1.5$

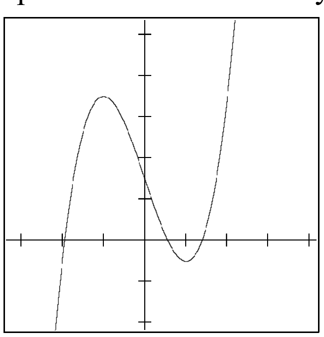


fig.45 $y=x^3-3x+1.5$

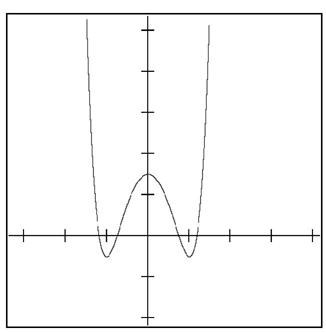


fig.46 $y=(x^2)^3-3x^2+1.5$

We do note that the y-intercept appears unchanged, and the y-axis is an axis of symmetry of D'which is an even function. Let's examine further examples.

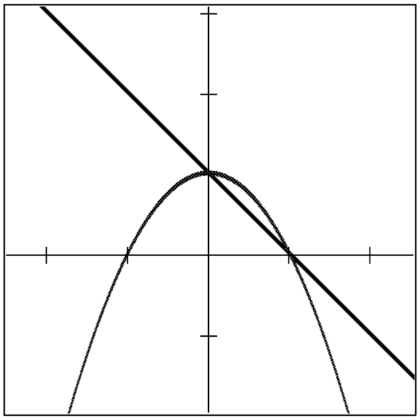


fig.47 $y=-x+1$ and $y=-x^2+1$(dotted)

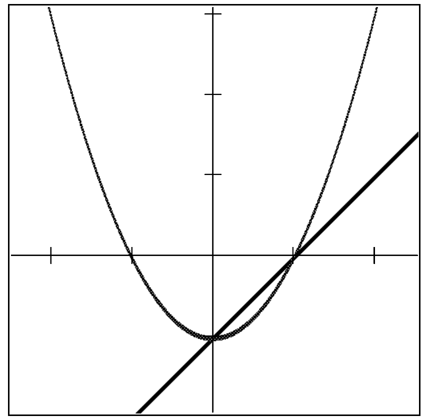


fig.48 $y=x-1$ and $y=x^2-1$(dotted)

We do note that $y=f(x)$ and $y=f(x^2)$ do share the y-intercept and there is symmetry over the y-axis. But a complete understanding may be elusive, so we examine further:

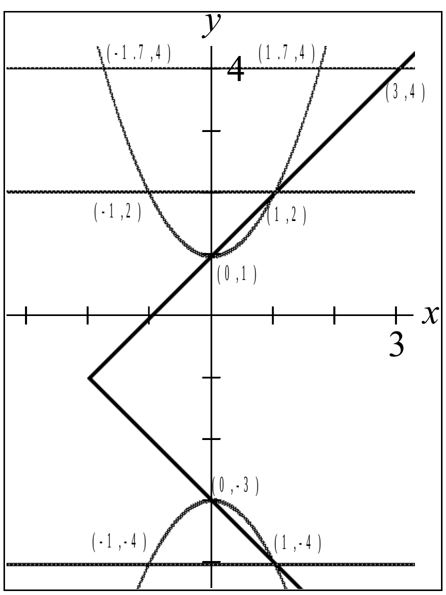


fig.49 R: $x=|y+1|-2$ and R': $x^2=|y+1|-2$ (dotted)

By examining the coordinates of specific points we can deduce: (a,b) is a point of R':R(x^2,y) if (a^2,b) is a point of R(x,y), where R is used to denote a relation. Or, put another way, if (a,b) is a point of R(x,y) then $(\pm\sqrt{a}, b)$ is a point of R(x^2,y). This clarifies why those points of R(x,y) in Quadrants 2 and 3 are discarded – their negative x-coordinates cannot be squares. The same conclusion may be expressed with respect to R(x,y) and R(|x|,y): for (a,b) on R(x,y) to generate a point (c,b) on R(|x|,y) then a must be the absolute value of c, implying that only Quadrant 1&4 points of R(x,y) qualify.

We are now at the stage of hypothesizing about such substitutions in general terms. Suppose we wish to graph R':R(f(x),y), given the graph of R(x,y).

(a,b) is a point of R': **R(f(x),y),**
iff (f(a),b) is a point of R: **R(x,y).**

Suppose we start with E: y=3x-1 and wish to graph E': y=3sinx-1.
(a,b) will be on E' iff (sina,b) is on E. Which implies that b, the y-coordinate, is 1 less than triple the sine of the first coordinate. I.e. b=3sina-1. Since –1 <sina <1 then b must lie between 3(-1)-1 and 3(1)-1, that is, between –4 and 2.

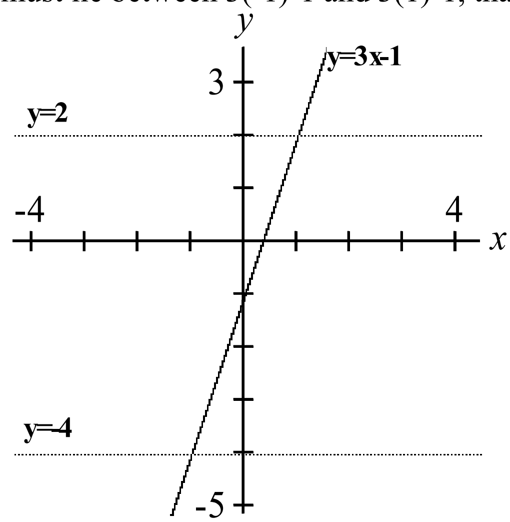


fig.50 E': y=3sinx-1 lies between the lines y=2 and y=-4

Now we can examine points of the segment of y=3x-1 between (1,2) and (-1,-4) to generate the corresponding points of y=3sinx-1.

E (0,-1) will generate ($\sin^{-1}(0)$,-1) on E': (0,-1), ($n\pi$,-1) where n is an integer
E (1,2) will generate ($\sin^{-1}(1)$,2) on E': ($\pi/2+2n\pi$,2) where n is an integer.

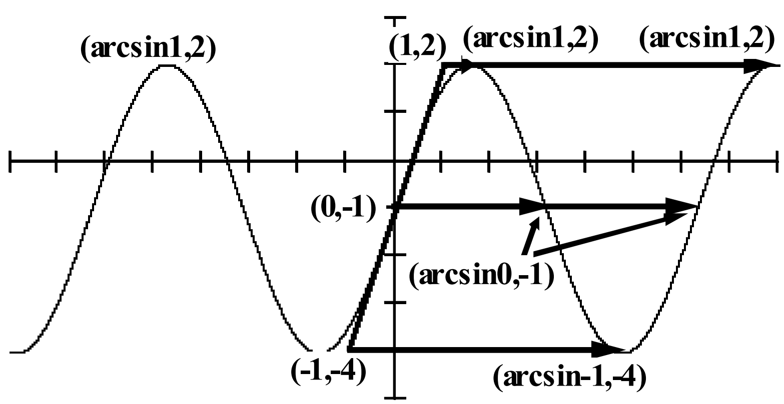


fig.51 Graph of y=3sinx-1 derived from the graph of y=3x-1

We can further generalize:

(a,b) on R(x,y)
will yield (f^{-1}(a),b) on R(f(x),y)

This concluding property will have a parallel equivalent version for y-replacements.

We can make further observations:

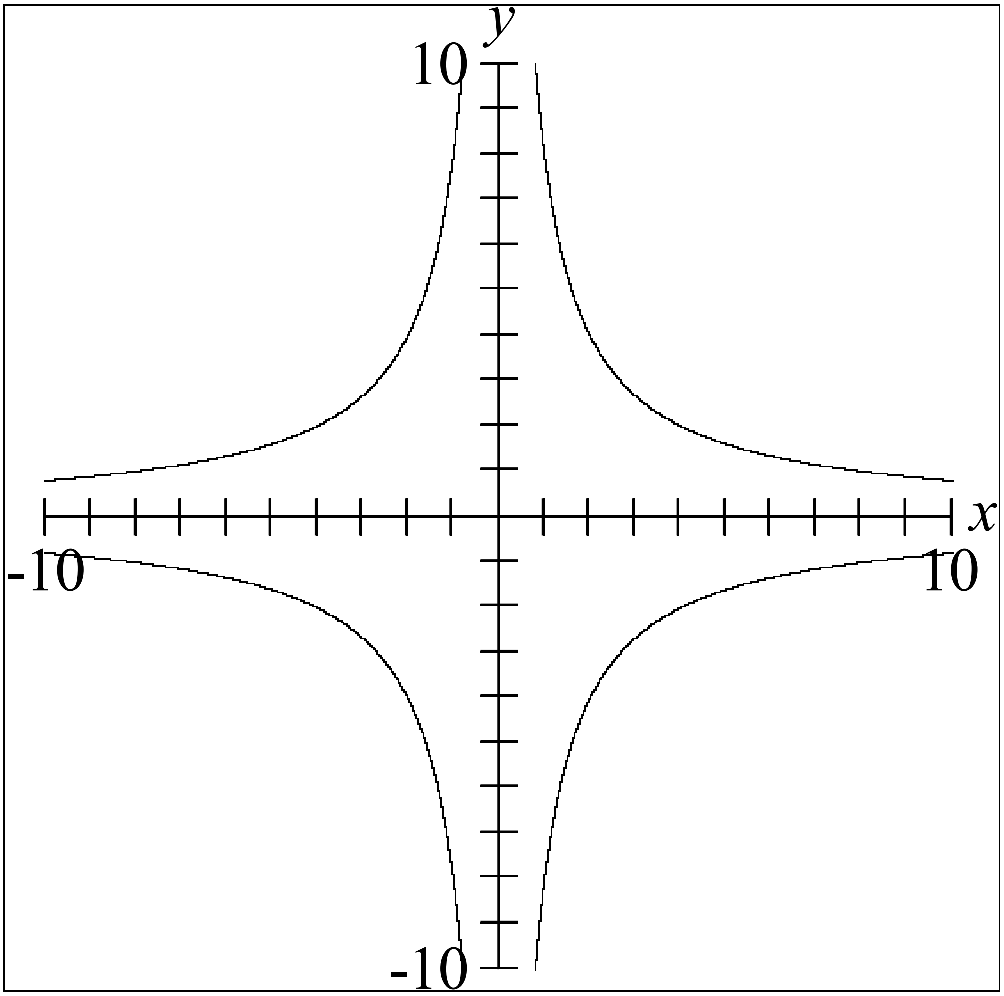
- Although the computer generally allows a direct access to the desired graph without the need for any analysis, it may also be used as a tool to gain analytic insights.
- The ‘scientific/experimental’ approach may be somewhat messy – there may be dead ends or ambiguous conclusions. There may be digressions. Not exactly “efficient”
- There may be an inclination on the teacher’s part to spare the students by simply stating the concluding property and reinforce it by examining sample cases. This will regrettably reinforce the notion that mathematics is simply a collection of procedures to be practiced.

Can we offer a decent example as to why the effect is the *opposite* of what might be expected? Consider the statement “The number of children we have is equal to the number of my month of birth (Mar. =3)”. If we replace *my month of birth* with *my wife’s month of birth* (June : 3 ***more than*** mine), then our statement becomes The number of children we have is 3 ***less than*** the number of my wife’s month of birth.”

Or, in mathematics parlance: given a function f: y=f(x) such that (a,b) is a solution, we can say that f maps a to b, or f(a)=b. If we replace x with g(x), then f will map g(x) to b provided g(x) = a. If g maps c to a (i.e. c = g^{-1}(a)) then we can conclude that (c,b) will be a solution of the composite function F: y=f(g(x)).

Conclusion: The foregoing presentation is intended to give some insight into an experimental approach to mathematical discovery. Including its false leads, incorrect conclusions and general inefficiency. With the obvious exception of the study of Euclidian geometry, most students are presented with mathematics as a *finished product*. They spend most of their time mastering taught techniques to solve a well-defined set of problems. The skill required boils down to two steps: first identify the type of problem, then recall and apply the appropriate solution procedure. In contrast, an experimental approach requires that the student attempt to *figure out* what is happening, and thereby gain some understanding.

Exercise: Given the graph of x^2y^2=64, *using the methods above*, sketch the graph of $(1/x)^2y^2$=64



$x^2y^2=64$