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Minkowski Geometric Algebra of Complex Sets

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Abstract. A geometric algebra of point sets in the complex plane is proposed, based on two fundamental operations: Minkowski sums and products. Although the (vector) Minkowski sum is widely known, the Minkowski product of two-dimensional sets (induced by the multiplication rule for complex numbers) has not previously attracted much attention. Many interesting applications, interpretations, and connections arise from the geometric algebra based on these operations. Minkowski products with lines and circles are intimately related to problems of wavefront reflection or refraction in geometrical optics. The Minkowski algebra is also the natural extension, to complex numbers, of interval-arithmetic methods for monitoring propagation of errors or uncertainties in real-number computations. The Minkowski sums and products offer basic 'shape operators' for applications such as computer-aided design and mathematical morphology, and may also prove useful in other contexts where complex variables play a fundamental role – Fourier analysis, conformal mapping, stability of control systems, etc.

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1. Introduction

The term *geometric algebra* has been employed in diverse contexts [1, 30, 40], but is currently most often associated with complex numbers, quaternions, and Clifford and Grassmann algebras. Informally, we may consider any space whose elements are subject to sum and product operations as constituting a geometric algebra, if the operations admit simple geometrical interpretations. Thus the first geometric algebra was probably the practice, in ancient Greece, of regarding products of two and three numbers as areas and volumes.

In this paper we propose a new geometric algebra with sums and products that admit an especially attractive and accessible geometrical interpretation. The space that interests us here is the *power set* $2^{\mathbb{C}}$ of the complex numbers $\mathbb{C}$ – i.e., *the set of all subsets of* $\mathbb{C}$. The sum and product operations on this space are the *Minkowski sum* $\oplus$ and *Minkowski product* $\otimes$, whose results are the subsets of $\mathbb{C}$ populated by the point-wise complex sums and products of all pairs of members drawn from their two complex-set operands.

There are no essential restrictions on the nature of the complex sets that are elements of this *Minkowski geometric algebra*: they may comprise discrete points, loci or regions in the complex plane, fractal sets, or any combination of these forms. To begin, however, we restrict our attention to simple regular sets (loci or regions) as the Minkowski sum or product operands. In addition to explicitly-defined sets, we wish to accommodate certain *implicitly*-defined sets (sets defined in a procedural manner, from which their geometrical nature is not immediately apparent) within the scope of this geometric algebra; such sets arise naturally in a variety of contexts and applications.

Our plan for this paper is as follows. In Section 2 we present the basic definitions and properties of Minkowski sums and products, and we motivate their study in Section 3 by discussing various applications, interpretations, and connections to other disciplines. The specification of 'explicit' and 'implicit' complex sets is then addressed in Section 4. Since Minkowski sums have already been extensively studied, we discuss them only briefly in Section 5 before proceeding to Minkowski products in Section 6, wherein several closed-form results for products with 'simple' operands are presented. Minkowski division can be cast as multiplication by an 'inverse' set, and hence in Section 7 we discuss the inversion of sets and Möbius transformations. A geometrical criterion that facilitates boundary evaluation for Minkowski products of general (smooth) operands is then identified in Section 8. Finally, Section 9 suggests some promising avenues for further investigation.

2. Geometric Algebra of Complex Sets

Let $\mathcal{A}$ and $\mathcal{B}$ denote point sets in the complex plane. No essential assumptions concerning the connectedness or dimensionality of these sets are required – they may comprise discrete points, loci, regions, or any combination thereof. The two fundamental operations that concern us are the *Minkowski sum* and *Minkowski product* of such sets, defined by*

$$\begin{aligned} \mathcal{A} \oplus \mathcal{B} &= \{\, \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B} \,\}, \\ \mathcal{A} \otimes \mathcal{B} &= \{\, \mathbf{a} \times \mathbf{b} \mid \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B} \,\}, \end{aligned} \tag{1}$$

where $+$ and $\times$ are the usual complex-number sum and product – namely, if $\mathbf{a} = a + \mathrm{i}\,\alpha$ and $\mathbf{b} = b + \mathrm{i}\,\beta$, we have

$$\mathbf{a} + \mathbf{b} = (a + b) + \mathrm{i}\,(\alpha + \beta) \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = (ab - \alpha\beta) + \mathrm{i}\,(a\beta + b\alpha)\,.$$

The Minkowski sum operation was introduced by Hermann Minkowski [42] in 1903 and has recently enjoyed resurgent interest, in the context of algorithms for geometric design, computer graphics, image processing, and related fields [22, 27, 32–34, 41, 54]. Of course, complex-number addition is equivalent to vector summation in

*Throughout this paper we denote real variables by italic characters, complex variables by bold characters, and sets of complex numbers by uppercase calligraphic characters.

$\mathbb{R}^2$, and the operation $\oplus$ is easily generalized to point sets in $\mathbb{R}^n$ by interpreting '+' as the appropriate vector sum.

On the other hand, the Minkowski product operation $\otimes$ has not previously (to the best of our knowledge) been systematically investigated. Although it is particular to the complex plane – or, equivalently, to $\mathbb{R}^2$ – we argue that the geometric algebra defined by the two operations (1) offers a remarkably appealing, fertile, and useful field of study. It provides a unifying framework for the description of geometrical operations (offsets, medial axis transforms, shape operators, etc.) that have formerly been treated as disparate functions; it furnishes a theoretical foundation for extending the well-known methods of (real) interval arithmetic to complex-number computations that incorporate 'uncertainty' information; and it yields remarkably elegant characterizations of key constructs (caustics & anticaustics) in classical geometrical optics. We feel sure that this brief catalog of insights, connections, and applications for the Minkowski geometric algebra (1) will be greatly enriched as its theoretical investigation unfolds and computational algorithms are elaborated.

From definitions (1) it is clear that the Minkowski operations $\oplus$ and $\otimes$ are commutative and associative, but in general we have

$$(\mathcal{A} \oplus \mathcal{B}) \otimes \mathcal{C} \neq (\mathcal{A} \otimes \mathcal{C}) \oplus (\mathcal{B} \otimes \mathcal{C}), \tag{2}$$

i.e., the distributive law does *not* hold. This can be seen by noting that the definitions of the sets in (2) can be reduced to

$$\begin{aligned} (\mathcal{A} \oplus \mathcal{B}) \otimes \mathcal{C} &= \{ \mathbf{ax} + \mathbf{bx} \mid \mathbf{a} \in \mathcal{A},\ \mathbf{b} \in \mathcal{B},\ \mathbf{x} \in \mathcal{C} \}, \\ (\mathcal{A} \otimes \mathcal{C}) \oplus (\mathcal{B} \otimes \mathcal{C}) &= \{ \mathbf{ax} + \mathbf{by} \mid \mathbf{a} \in \mathcal{A},\ \mathbf{b} \in \mathcal{B},\ \mathbf{x} \in \mathcal{C},\ \mathbf{y} \in \mathcal{C} \}. \end{aligned}$$

The first set comprises the complex values that are obtained when we *choose a single member of* $\mathcal{C}$, multiply it by arbitrary members of $\mathcal{A}$ and $\mathcal{B}$, and add the products. In the second set, on the other hand, we *independently choose two members of* $\mathcal{C}$, multiply them by arbitrary members from $\mathcal{A}$ and $\mathcal{B}$, and add the products. The first set is thus, in general, a *subset* of the second set, and hence we have the *subdistributive* law:

$$(\mathcal{A} \oplus \mathcal{B}) \otimes \mathcal{C} \subset (\mathcal{A} \otimes \mathcal{C}) \oplus (\mathcal{B} \otimes \mathcal{C}).$$

The Minkowski geometric algebra has a unique additive identity element (the set $\mathcal{O}$ comprising the single value 0) and multiplicative identity element (the set $\mathcal{I}$ comprising the single value 1). From definitions (1), however, one can easily see that a set $\mathcal{A}$ does *not* have an additive or multiplicative inverse, except in the trivial case that $\mathcal{A}$ comprises a single complex value.

Correspondingly, while the definitions (1) can be readily modified to also define Minkowski difference and division operations

$$\begin{aligned} \mathcal{A} \ominus \mathcal{B} &= \{ \mathbf{a} - \mathbf{b} \mid \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B} \}, \\ \mathcal{A} \oslash \mathcal{B} &= \{ \mathbf{a} \div \mathbf{b} \mid \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B} \}, \end{aligned} \tag{3}$$

(where one must ensure that $0 \notin \mathcal{B}$ if the division is to yield a bounded set), we cannot regard $\ominus$ and $\oslash$ as inverse operations to $\oplus$ and $\otimes$ since, in general,

$$(\mathcal{A} \oplus \mathcal{B}) \ominus \mathcal{B} \neq \mathcal{A} \qquad \text{and} \qquad (\mathcal{A} \otimes \mathcal{B}) \oslash \mathcal{B} \neq \mathcal{A}\,.$$

Actually, the operations (3) do not really offer any new functionality, since we can write $\mathcal{A} \ominus \mathcal{B} = \mathcal{A} \oplus (-\mathcal{B})$ and $\mathcal{A} \oslash \mathcal{B} = \mathcal{A} \otimes \mathcal{B}^{-1}$ instead, where

$$-\mathcal{B} = \{\, -\mathbf{b} \mid \mathbf{b} \in \mathcal{B}\,\} \qquad \text{and} \qquad \mathcal{B}^{-1} = \{\, \mathbf{b}^{-1} \mid \mathbf{b} \in \mathcal{B}\,\}$$

define the negation $-\mathcal{B}$ and reciprocal $\mathcal{B}^{-1}$ of any complex set $\mathcal{B}$. Thus, we shall henceforth employ only the operations $\oplus$ and $\otimes$.

As we shall see in Section 3, the geometric algebra of complex point sets, defined by the two Minkowski operations $\oplus$ and $\otimes$, is an attractive and fertile field of investigation, with extensive connections to classical geometry, and diverse potential applications. It is thus surprising that this subject is conspicuously absent from standard texts on complex analysis – even those that profess an overtly 'geometrical' perspective, such as Deaux [9], Schwerdtfeger [52], and Needham's beautifully-illustrated *Visual Complex Analysis* [47].

3. Applications, Connections, Interpretations

To motivate our investigation of the geometric algebra of complex point sets, we begin by briefly indicating some potential applications, connections, and interpretations. These encompass a generalization of real interval arithmetic to the complex domain, reflection and refraction of wavefronts in geometrical optics, stability characterization of multi-parameter control systems, and the shape analysis and procedural generation of two-dimensional domains. We expect that many other applications will become apparent as algorithms for practical computations with complex point sets are developed.

3.1. GENERALIZATION OF INTERVAL ARITHMETIC

Interval arithmetic is a formal algebra that provides the capability to monitor propagation of errors or uncertainties in real-variable computations [44, 45]. Intervals can be combined, according to prescribed arithmetic rules, to yield new intervals. These rules also allow us to define interval-valued *functions* of interval variables. Hence, standard algorithms, such as the Newton–Raphson root-finding method [25, 26], admit fairly straightforward generalizations to the interval context. The intervals in such computations may describe initial 'measurement uncertainties' in the input parameters to a problem, and also the effects of rounding errors (if interval endpoints are computed in floating-point arithmetic) by an extension known as *rounded interval arithmetic*.

By an interval $[a, b]$ we mean a set of real values of the form

$$[a, b] = \{ t \mid a \leqslant t \leqslant b \} . \tag{4}$$

It is understood that a variable x represented by the interval $[a, b]$ assumes any value between a and b with equal probability. Thus, variables known with certainty have definite real values – which can be interpreted as 'degenerate' intervals, of the form $a = [a, a]$.

Given two intervals $[a, b]$ and $[c, d]$, the result of an arithmetic operation $\star \in \{+, -, \times, \div\}$ on them is defined to be the set of all real values obtained by applying $\star$ to operands drawn from each interval [44]:

$$[a, b] \star [c, d] = \{ x \star y \mid x \in [a, b] \quad \text{and} \quad y \in [c, d] \} . \tag{5}$$

Specifically, one can easily verify that

$$\begin{aligned} &[a, b] + [c, d] = [a + c, b + d], \\ &[a, b] - [c, d] = [a - d, b - c], \\ &[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)], \\ &[a, b] \div [c, d] = [a, b] \times [1/d, 1/c], \end{aligned} \tag{6}$$

where division is usually defined only for denominators such that $0 \notin [c, d]$. Thus, for example,

$$[a, b] + [a, b] = [2a, 2b] \quad \text{and} \quad [a, b] - [a, b] = [a - b, b - a] .$$

Comparing the arithmetic of real numbers and of real intervals, certain common and distinct features are noteworthy. One can verify from (6) that interval addition and multiplication are both commutative and associative, but multiplication does *not* (in general) distribute over addition. The interval system has a unique additive identity $[0, 0] = 0$ and multiplicative identity $[1, 1] = 1$. However, an interval $[a, b]$ cannot possess an additive inverse or a multiplicative inverse unless it is degenerate – i.e., $a = b$.

The methods of interval arithmetic have been employed in algorithms for computer-aided design and computer graphics [46]. For example, the basic geometric primitives used in these algorithms, such as Bézier curves [12], can be generalized to the case where the control points are not specified precisely by real coordinate values, but rather by 'multi-dimensional intervals' – in the simplest case this means rectangular boxes [53], but the case of circular disks also admits a fairly straightforward treatment [35].

The geometric algebra defined by (1) offers a natural generalization from the arithmetic of real intervals to the arithmetic of compact simply-connected sets in the complex plane of arbitrary shape. Of course, 'simple' sets (disks or rectangles) are subsumed as special cases within the general theory.

Complex-number computations are crucial in many scientific/engineering applications – Fourier analysis, quantum mechanics, control systems, etc. – and the ability to perform these computations upon *sets* of complex numbers, not just discrete values, could have far-reaching implications. One may also formulate a theory of *complex set-valued functions* of complex sets. Another possibility is to define a real-valued, nonnegative *density function* $f(\mathbf{a})$ over the points $\mathbf{a} \in \mathcal{A}$ of a complex set. The composition of such functions, within the Minkowski geometric algebra, provides a more sophisticated probabilistic model for error propagation in complex-variable computations.

3.2. GEOMETRICAL OPTICS CONSTRUCTIONS

Minkowski products have some surprising and elegant connections to a basic construct of classical geometrical optics – the *anticaustic* for the reflection or refraction of spherical waves by a smooth surface. This correspondence is addressed thoroughly in Section 6 below – for the present, we shall confine ourselves to a qualitative description of its significance; see also [7, 14, 15].

The propagation of wavefronts in a homogeneous medium is described by *Huygens' principle* [56]. This states that, given an 'initial' wavefront W_0 at time 0, the propagated wavefront W at each subsequent time t is an *offset* or 'parallel' to W_0, at distance $d = ct$ from it (c is the wavespeed). Now in the presence of a smooth refracting or reflecting surface between different media, the wavefronts before and after the reflection or refraction are not members of a *single family* of offset surfaces. Nevertheless, we may still invoke Huygens' principle to characterize the reflected/refracted wavefronts as follows:

Suppose a spherical wave emanates from a point source at time $t = 0$ and, after reflection or refraction at a smooth surface $\mathcal{A}$ between two homogeneous media, subsequently assumes shape W at time t. By propagating *W backward in time in a single homogeneous medium*, we obtain a (hypothetical) 'initial' wavefront W_0 at $t = 0$. The significance of W_0 is that its uniform propagation via Huygens' principle (without reflection/refraction by the surface $\mathcal{A}$) yields the true reflected/refracted wavefront W at the prescribed time t.

The hypothetical 'initial' wavefront W_0, first studied by Jakob Bernoulli [2], is called⋆ the *anticaustic* for reflection or refraction of a spherical wave by the surface $\mathcal{A}$. The name anticaustic arises from the fact that W_0 is actually an involute of the *caustic* – i.e., the envelope of the reflected/refracted rays (which are normals to the reflected/refracted wavefronts). The 'caustic' – from the Greek for 'burning' – was thus named by Ehrenfried Walther von Tschirnhaus. Figure 1 illustrates the concept of the anticaustic.

For axisymmetric configurations of the light source and the surface $\mathcal{A}$, it suffices to restrict the problem to a plane of symmetry. The anticaustic then has a simple description in terms of our geometric algebra: it is the boundary $\partial(\mathcal{A} \otimes \mathcal{C})$ of the Minkowski product of a circle $\mathcal{C}$ and (a medial section of) $\mathcal{A}$. Examples of these

⋆The anticaustic appears under a variety of alternate names in the geometrical optics literature – the *secondary caustic* [6, 51], *orthotomic* [29], and *archetypal wavefront* [56].

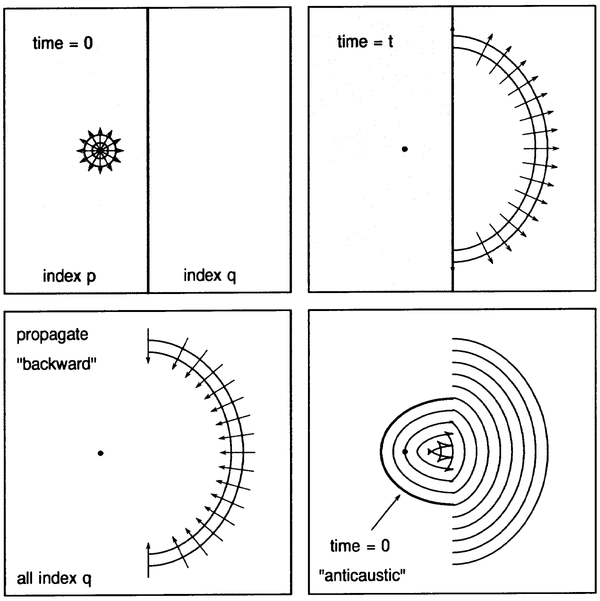

Figure 1. Definition of the anticaustic (an ellipse) for refraction of spherical waves by a planar interface between media with refractive indices p and q.

anticaustics are: an ellipse/hyperbola for refraction by a plane; a limaçon of Pascal for reflection by a sphere; and a Cartesian oval for refraction by a sphere. We elaborate on these results in Section 6 below.

3.3. STABILITY OF FEEDBACK CONTROL SYSTEMS

The *root locus method* [10, 31] is a standard means of analyzing the stability of linear feedback control systems. In the Laplace transform variable $\mathbf{s}$, let

$$c_n\mathbf{s}^n + \cdots + c_1\mathbf{s} + c_0 = 0 \tag{7}$$

be the *characteristic equation* of a control system. The roots of this equation are poles of the system transfer function, and for stability they must all have negative real parts. Now if the (real) coefficients $c_0, \ldots, c_n$ depend on a single (real) parameter k, the roots of (7) will trace out paths in the complex plane as k varies. These paths comprise the root locus of the control system, and one is interested in determining an admissible range of k values for which the loci of the roots of (7) lie entirely to the left of the imaginary axis.

The sole parameter k is usually the 'open loop gain' of the control system: the coefficients $c_0, \ldots, c_n$ depend linearly on it, and graphical rules [10, 31] can be used to qualitatively assess the geometrical form of the root loci. In certain contexts, however, it may be advantageous or necessary to analyze stability with respect to *several* (real) control parameters $k_1, \ldots, k_r$. If we imagine the coefficients of (7) to be dependent upon $r \geqslant 2$ parameters, its roots may cover a set of *regions* (not just *loci*) in the complex plane as each parameter k_i varies independently over some allowed interval $[a_i, b_i]$.

Thus, for a given set of (independent) parameter variations $k_i \in [a_i, b_i]$ for $1 \leqslant i \leqslant r$, and coefficients $c_j(k_1, \ldots, k_r)$ for $j = 0, \ldots, n$ of the characteristic equation (7) dependent on them, we are led to consider point sets of the form

$$\mathcal{R} = \left\{ \mathbf{s} \in \mathbb{C} \;\middle|\; \sum_{j=0}^{n} c_j(k_1, \ldots, k_r)\mathbf{s}^j = 0 \quad \text{for } k_i \in [a_i, b_i],\ 1 \leqslant i \leqslant r \right\} \tag{8}$$

in the complex plane. We call such a point set the *root domain* for the given characteristic equation coefficients and parameter variations, and the system is stable for any combination of parameters $k_1, \ldots, k_r$ in the specified ranges if the root domain $\mathcal{R}$ lies entirely to the left of the imaginary axis.

As a simple example, consider the 2-parameter quadratic equation

$$\mathbf{s}^2 + 2k_1\mathbf{s} + k_2 = 0 \quad \text{with } k_1, k_2 \in [0, 1]. \tag{9}$$

For pairs $k_1, k_2 \in [0, 1]$ with $k_1^2 \geqslant k_2$, both roots are real and they cover the interval $[-2, 0]$ as k_1, k_2 vary. When $k_1^2 < k_2$, on the other hand, the roots are complex conjugates and they cover the half-disk defined by $|\mathbf{s}| \leqslant 1$ and $\mathrm{Re}(\mathbf{s}) \leqslant 0$. Thus, for this system, the root domain $\mathcal{R}$ is of mixed dimension: the union of a one-dimensional locus (a real interval) and a two-dimensional area, as shown in Figure 2. The system is stable except for cases with $k_1 = 0$, in which both roots of equation (9) have $\mathrm{Re}(\mathbf{s}) = 0$.

The root domain (8) for a multi-parameter characteristic equation is an example of an *implicitly*-defined complex set – i.e., a set that is defined in a ‘procedural’ manner, from which its geometrical and topological properties are not immediately apparent: the boundary of such a set must be *computed*. An *explicitly*-defined complex set, on the other hand, is one whose geometry, topology, and boundary are directly evident from its definition.

Now sets such as the root domain (8) cannot, in general, be formulated as Minkowski combinations of ‘simple’ explicitly-defined sets. Nevertheless, we wish to include their analysis/evaluation within the scope of our geometric algebra, because of their fundamental importance in applications. When the explicit evaluation of an implicitly-defined complex set is difficult, it might be advantageous to invoke methods to approximate or contain that set by simpler sets, or Minkowski combinations of simpler sets.

One may generalize the definition (8) to allow coefficients $c_0, \ldots, c_n$ (and parameters $k_1, \ldots, k_r$) with complex values. For most applications, however, the coefficients (and the parameters they depend on) are real-valued. Since the roots of a polynomial with real coefficients are real or complex conjugate pairs, certain features of the simple root domain $\mathcal{R}$ for (9) shown in Figure 2 are generic to this

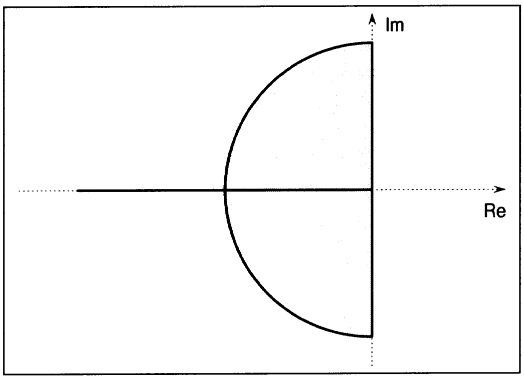

Figure 2. The root domain for the 2-parameter quadratic equation (9).

context – namely, it is the union of a set of real intervals and a set of complex regions that are symmetric about the real axis.

3.4. OFFSETS AND MEDIAL AXES OF PLANAR DOMAINS

The Minkowski geometric algebra offers a versatile medium for various shape construction and analysis functions that prove useful in applications such as geometric design, image processing, pattern recognition, and font generation. Many of these applications are still under active development – we confine ourselves here to mentioning a few representative examples.

A basic requirement of any computer-aided design system is the ability to compute the *offset* $\mathcal{A}_d$ at distance d to a planar domain $\mathcal{A}$ [50]. The offset domain $\mathcal{A}_d$ has a simple description in terms of Minkowski sums:

$$\mathcal{A}_d = \mathcal{A} \oplus \mathcal{S}_d ,$$

$\mathcal{S}_d$ being the disk of radius d centered on the origin. This defines an 'exterior' offset; we can also define an 'interior' offset by the expression

$$\mathcal{A}_{-d} = (\mathcal{A}^c \oplus \mathcal{S}_d)^c ,$$

where the superscript c denotes the complement of a set. These exterior and interior offset domains correspond to the results of the *dilation* and *erosion* operators, used in the field of *mathematical morphology* [54, 55].

In many applications, we are interested in the boundary of $\mathcal{A}_d$ – i.e., the *offset curve* to the boundary of $\mathcal{A}$. In computing offset curves, a fundamental difficulty arises from the fact that a rational curve does not, in general, have rational offsets. For example, the offset to a rational curve of degree n is [18] a (nonrational) algebraic curve⋆ of degree $6n - 4$ in general. Much effort has thus been devoted to the problem

⋆This curve describes the 'two–sided' offset, i.e., the offsets at distance $+d$ and $-d$.

of approximating offset curves; see [11] and references therein. The *Pythagorean-hodograph curves* are an exception – by construction, their unit normals depend rationally on the curve parameter, and hence their offsets are generically rational curves [13, 20, 48].

The *medial axis* – or 'skeleton' – of a planar domain $\mathcal{D}$ is the locus of centers of maximal disks (touching the boundary of $\mathcal{D}$ in at least two points) that may be inscribed within $\mathcal{D}$. By superposing a *radius function*, specifying the radius of the maximal inscribed disk at each point of the medial axis, we obtain the 'medial axis transform' (MAT) of the domain $\mathcal{D}$. The boundary of $\mathcal{D}$ can be precisely recovered from its MAT, as the envelope of the one-parameter family of these maximal inscribed disks.

Medial axis transforms have diverse applications in, for example, shape recognition and pattern analysis, image compression, path planning, surface fitting, font design, and mesh generation. The medial axis transform – and the closely-related *Voronoi diagram* [49] – are also very useful [8, 28] in the 'trimming' of a sequence of (untrimmed) offsets at successive distances d, which can otherwise be highly computation-intensive.

The process of boundary recovery from a MAT can be regarded as a form of 'scaled Minkowski sum' – if $\mathcal{A}$ and $\mathcal{B}$ are given complex sets, and f is a real-valued function defined on set $\mathcal{A}$, we say⋆ that

$$\mathcal{A} \oplus_f \mathcal{B} = \{\, \mathbf{a} + f(\mathbf{a})\mathbf{b} \mid \mathbf{a} \in \mathcal{A},\ \mathbf{b} \in \mathcal{B} \,\}$$

is the Minkowski sum of $\mathcal{A}$ and $\mathcal{B}$ scaled by f. Thus, if $\mathcal{M}$ is the medial axis, r is the radius function on $\mathcal{M}$, and $\mathcal{S}$ is the unit disk, the domain $\mathcal{D}$ can be represented [33] as the Minkowski sum of $\mathcal{M}$ and $\mathcal{S}$ scaled by r:

$$\mathcal{D} = \mathcal{M} \oplus_r \mathcal{S} = \{\, \mathbf{a} + r(\mathbf{a})\mathbf{b} \mid \mathbf{a} \in \mathcal{M},\ \mathbf{b} \in \mathcal{S} \,\}\,.$$

The boundary of $\mathcal{D}$ is, in general, a nonrational locus even if the segments of the medial axis $\mathcal{M}$ and the radius function r are rational. Exceptionally, if the MAT is specified by *Minkowski Pythagorean-hodograph curves* [43], the domain boundary is guaranteed to comprise rational segments.

4. Specification of Complex Sets

The remainder of this paper is devoted to an exploration of the properties and construction of Minkowski combinations. Since the Minkowski sum operation has been thoroughly investigated, we give only a brief summary of its salient features in Section 5. Our primary focus is on Minkowski products: in Section 6 we develop closed-form results for cases with 'basic' operands (points, lines, and circles), while in Section 8 we identify a key geometrical condition that facilitates boundary evaluation for products with more general set operands.

⋆In this notation, the subscript on $\oplus$ denotes the scaling function (whose domain is the first operand) that should be applied to each point of the second operand.

There are many possible ways to specify point sets in the complex plane. Before embarking on a discussion of Minkowski set operations, we must first establish an appropriate means for specifying set operands, that is sufficiently versatile to meet the needs of various applications. In Section 3.3, for example, we distinguished between 'explicitly' and 'implicitly' defined sets, and cited the root domain (8) as an example of the latter that arises in stability analysis of control systems. Expression (8) is actually a rather complicated example – as a simpler implicit set, consider

$$\mathcal{C} = \{\, \mathbf{a}^2 + \mathbf{ab} \mid \mathbf{a} \in \mathcal{A},\ \mathbf{b} \in \mathcal{B} \,\}\,.$$

At first, we may be tempted to identify $\mathcal{C}$ with $(\mathcal{A} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{B})$. But this is actually incorrect, for the same reason as the failure (2) of the distributivity law. Whereas points in $(\mathcal{A} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{B})$ arise from *three* simultaneous and independent choices of members from $\mathcal{A}$, points in $\mathcal{C}$ involve choosing only *one* member at a time from $\mathcal{A}$. Hence, $\mathcal{C} \subset (\mathcal{A} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{B})$.

In general, for a (complex) polynomial $\mathbf{f}(\mathbf{a}, \mathbf{b}, \ldots)$ in complex variables from prescribed sets $\mathcal{A}, \mathcal{B}, \ldots$, if any variable appears *more than once** in $\mathbf{f}$, the set $\{\, \mathbf{f}(\mathbf{a}, \mathbf{b}, \ldots) \mid \mathbf{a} \in \mathcal{A},\ \mathbf{b} \in \mathcal{B}, \ldots \}$ is *not* equivalent to the Minkowski combination obtained from $\mathbf{f}$ by replacing $\mathbf{a}, \mathbf{b}, \ldots$ by $\mathcal{A}, \mathcal{B}, \ldots$ and sums and products by the operators $\oplus$ and $\otimes$. Minkowski combinations imply *complete independence* in the choice of complex values from their respective sets, but multiple appearances of a variable in an expression always refer to the *same* value, not different values freely selected from some parent set.

Because of their importance in applications, we regard the consideration of implicitly-defined sets such as these to be a key element of the Minkowski geometric algebra. In fact, our interest in this subject arose in attempting to characterize a complex set of this nature that one encounters in the problem of Hermite interpolation by Pythagorean-hodograph quintics [19].

As noted above, given an implicit set defined by a polynomial in several variables, with multiple occurrences of at least one, we can define a Minkowski combination that is a *superset* of the implicit set. An explicit evaluation of the implicit set – i.e., a complete description of its boundary – is, however, a far more challenging task in general. Thus, we defer a thorough treatment of this problem to a subsequent paper that will fully address the computational aspects of the Minkowski geometric algebra.

5. Minkowski Sums

The Minkowski sum of two points sets, first introduced by H. Minkowski [42] in 1903, is a classical concept that has been extensively studied in the field of integral geometry [24, 39]. More recently, there has been considerable interest in developing algorithms to evaluate Minkowski sums for applications in areas such as computer

*Of course, squares and higher powers of a variable count as multiple appearances.

graphics, computer aided design, computer vision, image processing, and robotics: see, for example, [22, 32–34, 38, 41, 54, 55].

Our familiarity with the ordinary vector algebra of $\mathbb{R}^n$ imparts an intuitive appreciation for the meaning of $\mathcal{A} \oplus \mathcal{B}$, namely, the union of translates of $\mathcal{B}$ by vectors from the origin to the points of $\mathcal{A}$ (or vice-versa). In particular, there is little difficulty in visualizing such sums in Euclidean spaces of any dimension n, and one easily sees that the geometrical nature of the set $\mathcal{A} \oplus \mathcal{B}$ is independent of the location of the sets $\mathcal{A}$ and $\mathcal{B}$ relative to the origin.

Since they are well established, we do not propose to give a detailed review of the theoretical properties and computational methods for Minkowski sums here (the reader may consult the references cited above). Rather, we simply wish to emphasize that the above 'intuitive' properties of Minkowski sums (translation invariance and extensibility to any number of dimensions) must be relinquished upon introducing the notion of a Minkowski product.

6. Minkowski Products

We now derive exact results for basic Minkowski products involving 'simple' operands – i.e., points, lines, and circles. In this context, the conics and a quartic curve called the *Cartesian oval* (and various special instances thereof) play a fundamental role. Furthermore, we shall see that such products are intimately connected to certain classical problems of geometrical optics. The cases treated below exhaust the range of Minkowski products with tractable closed-form solutions. In Section 8, we develop some basic principles that facilitate the (approximate) computation of more general products. Envelopes of families of plane curves play a key role in the analysis of Minkowski products with simple operands. If $C(\lambda)$ is a one-parameter family of curves, continuously dependent on a (real) parameter λ, there are several approaches to defining its envelope. Three common definitions are:

- the envelope $\mathcal{E}$ is a plane curve that is tangent, at each of its points, to *some* curve in the family $C(\lambda)$;
- the envelope $\mathcal{E}$ is the locus, as λ varies, of the intersection points of 'neighboring' curves $C(\lambda)$ and $C(\lambda + \Delta\lambda)$, in the limit $\Delta\lambda \to 0$;
- if S is the surface obtained by 'stacking' each curve $C(\lambda)$ at height $z = \lambda$ above the (x, y) plane, the envelope $\mathcal{E}$ is the projection of the *silhouette* of S (as viewed along the z-axis) onto the (x, y) plane.

These definitions are not always precisely equivalent, and may be subject to certain technical qualifications under exceptional circumstances. We do not wish to be diverted into the technical details of envelope specifications here; the reader may consult [3–5, 16, 21] for a more detailed treatment. These problems require us to introduce qualifications into the statements of some of the results derived below – e.g., Propositions 3 and 6.

6.1. MULTIPLICATION BY POINTS

Suppose one of the operands in the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ is a *singleton* – i.e., a one-point set. Since the Minkowski product operation is commutative, we may assume without loss of generality that $\mathcal{A}$ is the set comprising just one (nonzero) complex point, $\mathbf{z} = |\mathbf{z}|\, e^{i\theta}$. The Minkowski product is then a trivial operation: namely, rotation of the complex set $\mathcal{B}$ about the origin by the angle θ, and scaling of it by the magnification factor $|\mathbf{z}|$. Although this is a very elementary operation in the complex plane, it provides the foundation for subsequent more-complicated Minkowski products.

Note that the operation of multiplication by a singleton admits a unique inverse. As in the case of the interval arithmetic (Section 3.1), if one operand in the Minkowski product degenerates to a one-point set $\{\mathbf{z}\}$, the set $\{\mathbf{z}^{-1}\}$ is its multiplicative inverse in our geometric algebra (except when $\mathbf{z} = 0$).

We now describe (without proof) some basic properties of multiplication by a point – one may easily verify them. These properties will help facilitate subsequent derivations of more complicated products.

PROPOSITION 1. *If* $\mathbf{w}, \mathbf{z}$ *are fixed nonzero complex numbers and* $\mathcal{A}, \mathcal{B}$ *are point sets in the complex plane, the following properties hold*

$$\mathcal{A} = \{\mathbf{z}^{-1}\} \otimes \{\mathbf{z}\} \otimes \mathcal{A}\,, \tag{10}$$

$$\mathcal{A} \otimes \mathcal{B} = \{\mathbf{z}^{-1}\mathbf{w}^{-1}\} \otimes (\{\mathbf{z}\} \otimes \mathcal{A}) \otimes (\{\mathbf{w}\} \otimes \mathcal{B})\,. \tag{11}$$

Now the relation (11) allows us to perform certain 'normalizations' before computing a Minkowski product $\mathcal{A} \otimes \mathcal{B}$. Given sets $\mathcal{A}$ and $\mathcal{B}$, we first move⋆ both of them into 'standard' locations, by multiplying them individually by suitably-chosen complex numbers, $\mathbf{z}$ and $\mathbf{w}$. We then compute the Minkowski product of these 'normalized' sets. Finally, multiplying the resulting set by the inverses $\mathbf{z}^{-1}$ and $\mathbf{w}^{-1}$ yields the desired Minkowski product $\mathcal{A} \otimes \mathcal{B}$.

Multiplication by singleton sets offers a fruitful perspective on Minkowski products of general point sets. Namely, such products can be interpreted as the union of all sets that are obtained by multiplying the entirety of one set by each constituent point of the other set:

$$\mathcal{A} \otimes \mathcal{B} = \bigcup_{\mathbf{z}\in\mathcal{A}} \{\mathbf{z}\} \otimes \mathcal{B} = \bigcup_{\mathbf{z}\in\mathcal{B}} \mathcal{A} \otimes \{\mathbf{z}\}\,.$$

In particular, when $\mathcal{A}$ is a parameterized curve in the complex plane we can consider $\mathcal{A} \otimes \mathcal{B}$ to be the union of the one-parameter family of sets that are obtained by applying certain scalings and rotations to $\mathcal{B}$.

⋆Note that the word *move* here, meaning multiplication by a nonzero complex number, connotes a combination of scaling and rotation about the origin.

6.2. PRODUCT OF TWO LINES

As the first nontrivial example of a Minkowski product, we now show that multiplying two lines gives, in general, the region outside a parabola.

Let $\mathcal{A}$ and $\mathcal{B}$ be lines in the complex plane. To begin, we assume neither of them passes through the origin. Then, using (11), we can transform both of them into vertical lines passing through the point 1 on the real axis without loss of generality. Thus, it is sufficient to deal with the sets

$$\mathcal{A} = \{ 1 + \mathrm{i}t \mid t \in \mathbb{R} \}, \quad \mathcal{B} = \{ 1 + \mathrm{i}s \mid s \in \mathbb{R} \}.$$

We may consider the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ as the union of a one-parameter family of lines. Since multiplication by $1 + \mathrm{i}t$ transforms $\mathcal{B}$ into a line that passes through the point $1 + \mathrm{i}t$, and is perpendicular to the line connecting 0 and $1 + \mathrm{i}t$, the equation of this one-parameter family of lines is

$$\phi(x, y, t) = x + ty - t^2 - 1 = 0.$$

Invoking the usual procedure [3, 4] for envelope computations, we find upon eliminating t among the equations $\phi(x, y, t) = 0$ and $\partial\phi(x, y, t)/\partial t = 0$ that this family of lines has the parabola $y^2 = 4(1 - x)$ as envelope. The vertex of this parabola is at the point 1 on the real axis, and the focus is at the origin. Figure 3 illustrates the family of lines, and its envelope.

Thus, the Minkowski product of two lines $\mathcal{A}$ and $\mathcal{B}$ in 'standard location' is the region $\{ x + \mathrm{i}y \mid y^2 \geqslant 4(1 - x) \}$. Each point $\mathbf{z} = x + \mathrm{i}y$ in the interior of this region, i.e., $y^2 > 4(1 - x)$, is the product of *two* distinct pairs of points from $\mathcal{A}$ and $\mathcal{B}$. On the other hand, every point $\mathbf{z}$ on the boundary of the region is generated by a *unique* pair of points from $\mathcal{A}$ and $\mathcal{B}$. The Minkowski product of *any* pair of lines (not passing through the origin) can obtained by means of a suitable rotation and scaling of this region.

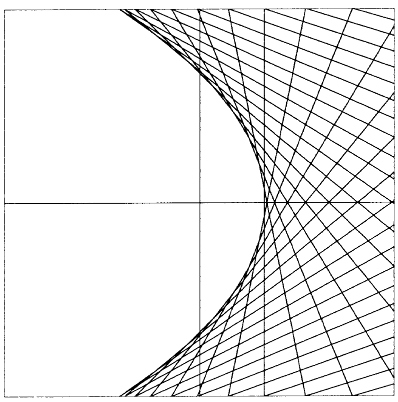

Figure 3. The Minkowski product of two lines.

Consider now the case where one of the two lines ($\mathcal{A}$, say) passes through the origin. Then we can normalize the two sets as follows:

$$\mathcal{A} = \{ t \mid t \in \mathbb{R} \}, \qquad \mathcal{B} = \{ 1 + \mathrm{i}s \mid s \in \mathbb{R} \}.$$

Each nonzero t on $\mathcal{A}$ transforms $\mathcal{B}$ into a vertical line passing through the value t on the real axis. The union of all such lines fills the entire complex plane, except the imaginary axis. And the point 0 on $\mathcal{A}$ shrinks $\mathcal{B}$ to a single point, at the origin. Hence, $\mathcal{A} \otimes \mathcal{B}$ is the set $\{ \mathbf{z} \mid \mathrm{Re}(\mathbf{z}) \neq 0 \} \cup \{ 0 \}$. In the case that both the lines $\mathcal{A}$ and $\mathcal{B}$ pass through the origin, we transform both into the real axis, and their Minkowski product is just the real line.

We now summarize these results for the Minkowski product of two lines, according to whether or not they pass through the origin:

PROPOSITION 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be lines in the complex plane. Then:*

(a) *when neither $\mathcal{A}$ nor $\mathcal{B}$ passes through the origin, the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ is the region outside of a parabola;*
(b) *when just one of $\mathcal{A}$ and $\mathcal{B}$ passes through the origin, $\mathcal{A} \otimes \mathcal{B}$ is the union of the origin and two half planes separated by a line through the origin;*
(c) *when both $\mathcal{A}$ and $\mathcal{B}$ pass through the origin, $\mathcal{A} \otimes \mathcal{B}$ is also a line passing through the origin.*

6.3. MULTIPLICATION BY LINES – NEGATIVE PEDALS

We have shown that, in general, the Minkowski product of lines $\mathcal{A}$ and $\mathcal{B}$ is the region bounded by a parabola. Suppose we now replace one of the lines ($\mathcal{A}$, say) by a smooth curve C in the complex plane. We will now show that the Minkowski product of C and a line that does not pass through the origin is closely related to the *negative pedal* of C with respect to the origin.

For a given plane curve C and fixed point $\mathbf{o}$, the *pedal* curve C' of C with respect to $\mathbf{o}$ is defined [36, 37] to be the locus of the foot of the perpendicular drawn from $\mathbf{o}$ to the tangent line of curve C at a point $\mathbf{p}$ that moves along it. Conversely, a curve C that has a given curve C' as its pedal with respect to $\mathbf{o}$ is called the *negative pedal* of C' with respect to $\mathbf{o}$ (the negative pedal is consequently the envelope of lines drawn through each point $\mathbf{q}$ of C', that are perpendicular to $\overline{\mathbf{oq}}$). Figure 4 illustrates these geometrical constructions.

Now if the line $\mathcal{B}$ does not pass through the origin, we may transform it into the vertical line passing through the point 1 on the real axis, as before. However, we do not impose any particular normalization on the operand $\mathcal{A}$ corresponding to the curve C in the complex plane.

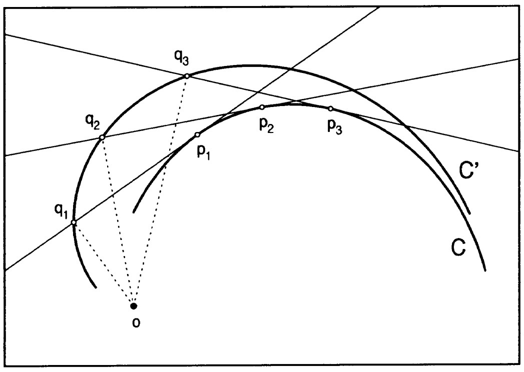

Figure 4. Points $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ on curve C' are footpoints of the perpendiculars (dashed lines) drawn from $\mathbf{o}$ to tangents of the curve C at points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ on it. Thus, C' is the pedal of C with respect to $\mathbf{o}$. Conversely, lines through $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ on C' that are perpendicular to the radii drawn from $\mathbf{o}$ are tangent to C at $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and C is thus the negative pedal of C' with respect to $\mathbf{o}$.

PROPOSITION 3. *Let $\mathcal{A}$ be a smooth curve C in the complex plane, and let $\mathcal{B}$ be the vertical line through 1. Then $\partial(\mathcal{A} \otimes \mathcal{B})$ is ordinarily (a subset of) the negative pedal of C with respect to the origin.*

Proof. We regard $\mathcal{A} \otimes \mathcal{B}$ as the union of a one-parameter family of lines – each point $\mathbf{z}$ of $\mathcal{A}$ transforms $\mathcal{B}$ into a line passing through $\mathbf{z}$, perpendicular to the line connecting $\mathbf{z}$ with the origin. The envelope of this one-parameter family of lines is the negative pedal of the curve C with respect to the origin. In 'simple' instances, the boundary $\partial(\mathcal{A} \otimes \mathcal{B})$ will be identical to the envelope of this family of lines. For a general curve $\mathcal{A}$, however, we must allow for the possibility that: (i) portions of the envelope lie in the *interior* of the region $\mathcal{A} \otimes \mathcal{B}$; and (ii) in exceptional circumstances, portions of individual lines in the family (which are not considered part of the envelope) may contribute to the boundary of $\mathcal{A} \otimes \mathcal{B}$. Thus, for a general curve $\mathcal{A}$ and a line $\mathcal{B}$, we qualify our identification of $\partial(\mathcal{A} \otimes \mathcal{B})$ with the negative pedal by saying 'ordinarily' and 'a subset of' in Proposition 3. □

Pedals and negative pedals have a special significance in geometric optics [29]. Suppose that the pedal point $\mathbf{o}$ and curve C represent a light source and a mirror. Since each point $\mathbf{q}$ on the pedal of C with respect to $\mathbf{o}$ is the foot of the perpendicular from $\mathbf{o}$ to a tangent line of C, we can obtain the anticaustic for reflection of spherical waves from $\mathbf{o}$ by C through a radial scaling of the pedal curve about $\mathbf{o}$ by a factor 2. Conversely, we can design the mirror that yields a given anticaustic C for reflection of spherical waves from $\mathbf{o}$, through a radial scaling about $\mathbf{o}$ by a factor $\frac{1}{2}$ of the negative pedal of C with respect to $\mathbf{o}$. Figure 5 illustrates the geometry of these problems, which admits another interpretation: each point of C is equidistant from the source point $\mathbf{o}$ and corresponding anticaustic point $\mathbf{r}$ – hence the mirror C is the (untrimmed) *point/curve bisector* [16] of the source and the anticaustic.

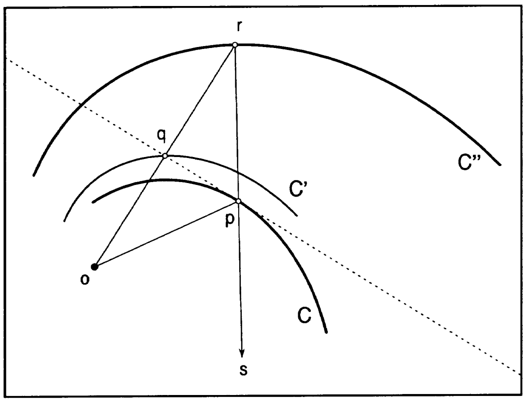

Figure 5. Anticaustic for reflection of spherical waves as a pedal curve: with light source **o** and mirror C, a ray from **o** is reflected at point **p** of C to point **s**. The dashed line is the tangent of C at **p**, and hence the footpoint **q** of the perpendicular from **o** to this line lies on the pedal C' of C with respect to **o**. Scaling C' radially about **o** by 2 yields C'', the triangles **o**, **p**, **q** and **r**, **p**, **q** being similar, where **r** on C'' corresponds to **q** on C'. Since $|\mathbf{p} - \mathbf{r}| = |\mathbf{p} - \mathbf{o}|$, C'' represents the anticaustic for reflection of spherical waves from **o** by C.

6.4. PRODUCT OF A LINE AND A CIRCLE

Suppose that $\mathcal{A}$ is a circle and $\mathcal{B}$ is a line. We first deal with some exceptional yet simple cases. If the line $\mathcal{B}$ passes through the origin, we may transform it into the real axis by a rotation. Then each **z** on the circle $\mathcal{A}$ transforms $\mathcal{B}$ into the line passing through **z** and the origin. Thus, if the origin is inside the circle, this family of lines will sweep out the entire complex plane. When the origin lies on the circle, the Minkowski product covers the entire plane *except* the circle tangent line at the origin (but the origin itself is included). Finally, when the origin is outside the circle, the family of lines fills the wedge-shaped region between the two tangent lines to the circle drawn from the origin.

Assuming henceforth that the line $\mathcal{B}$ does not go through the origin, we transform it into the vertical line passing through 1 on the real axis. Another exceptional case occurs when the center of $\mathcal{A}$ is the origin: one can easily see that the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ is the region outside the circle $\mathcal{A}$.

In the general case, we may assume that the center of $\mathcal{A}$ is 1 and $\mathcal{B}$ is the vertical line passing through 1. The Minkowski product $\mathcal{A} \otimes \mathcal{B}$ then has two interpretations. As in the preceding section, we may consider $\partial(\mathcal{A} \otimes \mathcal{B})$ to be the negative pedal of the circle $\mathcal{A}$ with respect to the origin. The negative pedal of a circle is an ellipse or a hyperbola, according to whether or not the circle contains the pedal point. Thus, the Minkowski product of the circle $\mathcal{A}$ and the line $\mathcal{B}$ is bounded by an ellipse or a hyperbola, according to whether the origin is inside or outside $\mathcal{A}$. Figure 6 illustrates the family of lines when the radius of the circle is 3/2 (on the left), and 3/4 (on the right).

The other interpretation of the Minkowski product of the circle $\mathcal{A}$ and the line $\mathcal{B}$ is the one-parameter family of circles generated by multiplying $\mathcal{A}$ by each point **z** on the

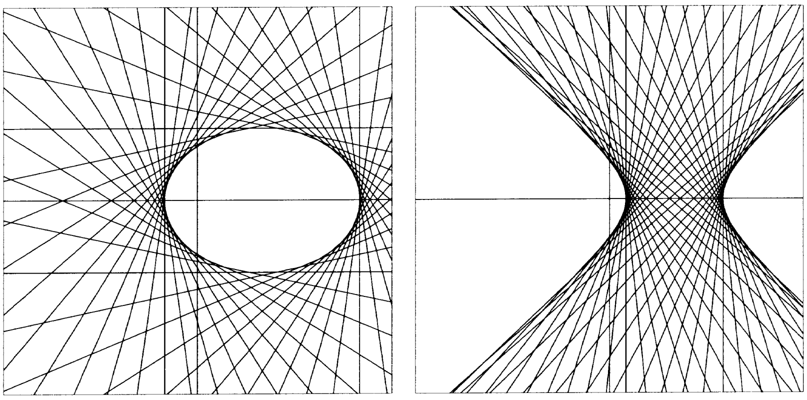

Figure 6. The Minkowski products of a line and a circle.

line $\mathcal{B}$. We now show that this interpretation yields exactly the same result for $\partial(\mathcal{A} \otimes \mathcal{B})$: an ellipse or a hyperbola.

PROPOSITION 4. *Let $\mathcal{A}$ be the circle with radius r and center 1, and let $\mathcal{B}$ be the vertical line through 1. The Minkowski product $\mathcal{A} \otimes \mathcal{B}$ is then as follows:*

(a) *if $r > 1$, $\mathcal{A} \otimes \mathcal{B}$ is the region outside an ellipse;*
(b) *if $r < 1$, $\mathcal{A} \otimes \mathcal{B}$ is the region between the branches of a hyperbola;*
(c) *if $r = 1$, $\mathcal{A} \otimes \mathcal{B}$ is the region defined by $\{\,\mathbf{z} \mid \mathrm{Im}(\mathbf{z}) \neq 0\,\} \cup \{\,0, 2\,\}$.*

Proof. Writing the operands $\mathcal{A}$ and $\mathcal{B}$ in the form

$$\mathcal{A} = \{\, x + \mathrm{i}y \mid (x-1)^2 + y^2 = r^2 \,\}, \qquad \mathcal{B} = \{\, 1 + \mathrm{i}t \mid t \in \mathbb{R} \,\},$$

the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ can be written as the union

$$\mathcal{A} \otimes \mathcal{B} = \bigcup_{t \in \mathbb{R}} \{1 + \mathrm{i}t\} \otimes \mathcal{A}.$$

Each point $1 + \mathrm{i}t$ of $\mathcal{B}$ transforms $\mathcal{A}$ into a circle with center $1 + \mathrm{i}t$ and radius $r|1 + \mathrm{i}t|$. So the one parameter family of circles is written in the form of

$$|x + \mathrm{i}y - (1 + \mathrm{i}t)| = r\,|1 + \mathrm{i}t|$$

or

$$\phi(x, y, t) = (x-1)^2 + (y-t)^2 - r^2(1 + t^2) = 0. \tag{12}$$

And the partial derivative of ϕ with respect to t is

$$\frac{\partial \phi}{\partial t}(x, y, t) = -2(y-t) - 2r^2 t.$$

By eliminating t among the equations $\phi = 0$ and $\partial\phi/\partial t = 0$, we obtain the envelope of this family of circles:

$$(x-1)^2 + \frac{r^2}{r^2-1}\, y^2 = r^2 . \tag{13}$$

Thus, $\partial(\mathcal{A} \otimes \mathcal{B})$ is an ellipse or a hyperbola according to whether r is greater than or less than 1. When $r = 1$, on the other hand, the family (12) consists of all circles passing through the point 2 and the origin. The union of these circles comprises all points with $\mathrm{Im}(z) \neq 0$, plus the real points 0 and 2. □

Figure 7 shows the same Minkowski line/circle products as in Figure 6, but interpreted in terms of one-parameter families of circles.

6.5. PRODUCT OF TWO CIRCLES

We now consider the Minkowski product of two circles, $\mathcal{A}$ and $\mathcal{B}$. In general, this is the region bounded by a curve known as the *Cartesian oval*.

We first deal with certain exceptional cases. If the centers of both circles are at the origin, the Minkowski product is also a circle centered at the origin whose radius is the product of the radii of $\mathcal{A}$ and $\mathcal{B}$. If only one of the circles ($\mathcal{A}$, say) has center at the origin, we can transform $\mathcal{A}$ into the unit circle in the complex plane, and $\mathcal{B}$ into a circle with center 1 and radius r. One can then easily see that the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ is the annular region defined by $|1-r| \leqslant |\mathbf{z}| \leqslant 1+r$. For circles in general position, we have:

PROPOSITION 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two circles with centers not at the origin. Then $\partial(\mathcal{A} \otimes \mathcal{B})$ is a Cartesian oval, and the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ is the region between the two loops of the Cartesian oval.*

Since the Cartesian oval is not a particularly well-known curve, we briefly review its definition and basic properties before proceeding with the proof of Proposition 5. Conceptually, the simplest description of a Cartesian oval is in terms of *bipolar coordinates* – i.e., the distances r_1 and r_2 of a point on the curve from two fixed 'poles' in the plane. Without loss of generality, we may take poles at $(0, 0)$ and $(a, 0)$. Then, for nonzero real values m and n, the Cartesian oval is described by the bipolar equation*

$$m r_1 \pm n r_2 = \pm 1 , \tag{14}$$

which subsumes the ellipse/hyperbola ($m = \pm n$) and circle ($a = 0$) as special cases. To describe the general Cartesian oval as an algebraic curve, we need to take squares

*In fact, a Cartesian oval admits three distinct bipolar descriptions [23]. We may choose any two of three possible poles, and for each pair there are corresponding m and n values.

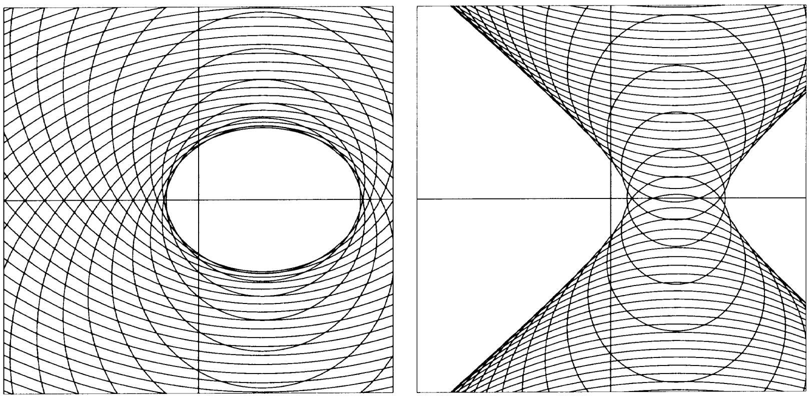

Figure 7. The families of circles defining the Minkowski products in Figure 6.

twice in (14) to clear radicals – this gives

$$(m^2 r_1^2 - n^2 r_2^2)^2 \; - \; 2\,(m^2 r_1^2 + n^2 r_2^2) \; + \; 1 \; = \; 0\,, \tag{15}$$

where $r_1^2 = x^2 + y^2$ and $r_2^2 = (x - a)^2 + y^2$. Thus, the general Cartesian oval is an algebraic curve of degree 4. It consists of two loops that comprise a single irreducible curve.⋆ It has double points at the circular points at infinity, but (except in degenerate cases) no other singularities, and is thus of genus 1.

The Cartesian oval is of fundamental importance in geometrical optics: it is the anticaustic for refraction of a spherical wavefront (from a point source) by a spherical surface. By symmetry, we need only consider a planar section containing the point source and the center of the refracting sphere. Suppose this circle has center 1 and radius r, and let p and q be the refractive indices associated with the interior and exterior of the circle. If the source is at the origin, the *optical path length* between the origin and a point $x + \mathrm{i}y$ outside the circle, via the point $1 + r\mathrm{e}^{\mathrm{i}\theta}$ on it, is defined by

$$\ell \; = \; p\,|\,1 + r\mathrm{e}^{\mathrm{i}\theta}\,| \; + \; q\,|\,x + \mathrm{i}y - (1 + r\mathrm{e}^{\mathrm{i}\theta})\,|\,.$$

On setting $\ell = 0$ and squaring, we obtain the one-parameter family of circles

$$k^2\,[\,(x - 1 - r\cos\theta)^2 + (y - r\sin\theta)^2\,] \; - \; (r^2 + 2r\cos\theta + 1) \; = \; 0\,, \tag{16}$$

where $k = q/p$. The anticaustic is, by definition, the envelope of this family of circles. We can express (16) rationally in terms of a parameter t by setting $\cos\theta = (1 - t^2)/(1 + t^2)$ and $\sin\theta = 2t/(1 + t^2)$. Eliminating t between the resulting

⋆Only two of the four possible sign combinations in equation (14) define real loci.

expression and its partial derivative with respect to t then gives the Cartesian oval equation*

$$[\,k^2((x-1)^2+y^2+r^2)-(r^2+1)\,]^2\;-\;4r^2[\,(k^2(x-1)+1)^2+k^4y^2\,]\;=\;0\,. \tag{17}$$

This equation is of the form (15), with $m=k/(1-k^2)r$ and $n=k^2/(1-k^2)r$, the distances r_1 and r_2 being measured from poles at $(0,0)$ and $(1-k^{-2},0)$.

We now show that the boundary of the Minkowski product of two circles is a Cartesian oval: the Minkowski product occupies the region between the two loops of the Cartesian oval.

Proof of Proposition 5. Since neither of the circles has center at the origin, we can transform both into circles with center 1. The operands $\mathcal{A}$ and $\mathcal{B}$ of the Minkowski product are then of the form

$$\mathcal{A}\;=\;\{\,1+r\,e^{i\theta}\mid 0\leqslant\theta<2\pi\,\}\,,\quad \mathcal{B}\;=\;\left\{\,1+\frac{1}{k}\,e^{i\psi}\mid 0\leqslant\psi<2\pi\,\right\}.$$

Now the Minkowski product $\mathcal{A}\otimes\mathcal{B}$ can be regarded as the union of a one-parameter family of circles, of the form

$$\mathcal{A}\otimes\mathcal{B}\;=\;\bigcup_{\theta}\{\,1+r\,e^{i\theta}\,\}\otimes\mathcal{B}\,. \tag{18}$$

Since multiplication by $1+r\,e^{i\theta}$ transforms $\mathcal{B}$ into a circle with center $1+r\,e^{i\theta}$ and radius $k^{-1}|\,1+r\,e^{i\theta}\,|$, the one-parameter family of circles in equation (18) is identical to that defined by equation (16). Therefore, the boundary of the Minkowski product is a Cartesian oval. Each circle in the family (18) touches both the inner and the outer loop of the Cartesian oval, and hence the Minkowski product $\mathcal{A}\otimes\mathcal{B}$ occupies the region between the two loops. □

The Minkowski product $\mathcal{A}\otimes\mathcal{B}$ can also be interpreted as the union of the family of circles obtained by multiplying $\mathcal{A}$ by each point of $\mathcal{B}$:

$$\mathcal{A}\otimes\mathcal{B}=\bigcup_{\psi}\mathcal{A}\otimes\left\{\,1+\frac{1}{k}\,e^{i\psi}\,\right\}. \tag{19}$$

Figure 8 illustrates the two one-parameter families of circles defined by (18) and (19). Although the two families of circles are different, they clearly have the same Cartesian oval as their envelopes. In fact, both are consistent with the definition of Cartesian ovals given by Gomes Teixeira [23, p. 233]:

> L'enveloppe d'un cercle variable dont le centre parcourt la circonférence d'un autre cercle donné et dont le rayon varie proportionnellement à la distance de son centre à un point

*In interpreting this as an anticaustic, we tacitly assume that the source lies *inside* the refracting sphere ($r>1$), and hence the entire wavefront suffers only a *single* refraction. If $r<1$, only a portion of the wavefront suffers refraction (in fact, it is refracted *twice* — first on entering the sphere, and subsequently on emerging from it).

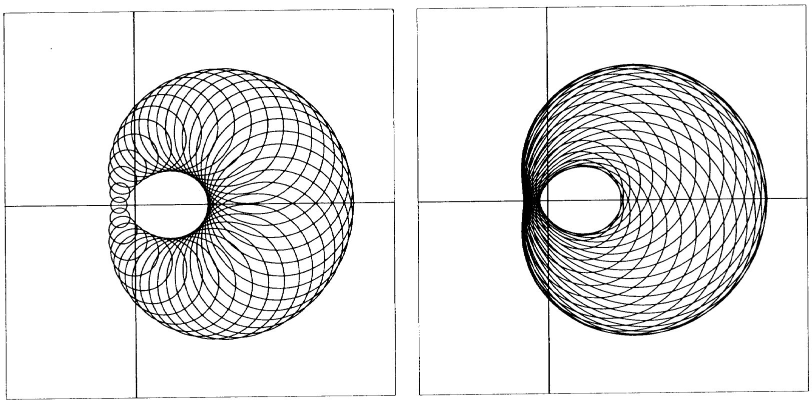

Figure 8. The two one-parameter families of circles (18) and (19), defining the Minkowski product of two circles (of radii 0.5 and 1.25) that do not pass through the origin, with the same Cartesian oval as their envelopes.

fixe est un couple d'ovales de Descartes.

We can now state a new and especially succinct definition: *a Cartesian oval is the boundary of the Minkowski product of two circles.*

Finally, we mention the noteworthy special case $k^2 = 1$. In this case, the Cartesian oval degenerates into a *limaçon of Pascal*,

$$[(x-1)^2 + y^2 - 1]^2 - 4r^2(x^2 + y^2) = 0,$$

which is the anticaustic for *reflection* of spherical waves by a spherical surface. The limaçon of Pascal is evidently the boundary of the Minkowski product of two circles, one of which passes through the origin. In addition to double points at the circular points at infinity, the limaçon also has an affine double point at the origin, and is thus a *rational* curve. The affine double point is a *crunode* (self-intersection) for $r < 1$, and an *acnode* (isolated real point) for $r > 1$. Figure 9 shows these two forms of limaçon, as the envelopes of families of circles. Exceptionally, when $r = 1$, both circles $\mathcal{A}$ and $\mathcal{B}$ in the Minkowski product pass through the origin, and the affine double point of the limaçon is a *cusp* – this form, known as the *cardioid*, is shown in Figure 10.

6.6. MULTIPLICATION BY CIRCLES – ANTICAUSTICS

We have seen above that the Minkowski product of two circles generates a region bounded by a Cartesian oval, which is the anticaustic for refraction of a spherical wave by a spherical surface. In their normalized descriptions, both circles have center 1, and radii r (the radius of the refracting sphere) and k^{-1} (where $k = q/p$ is the ratio

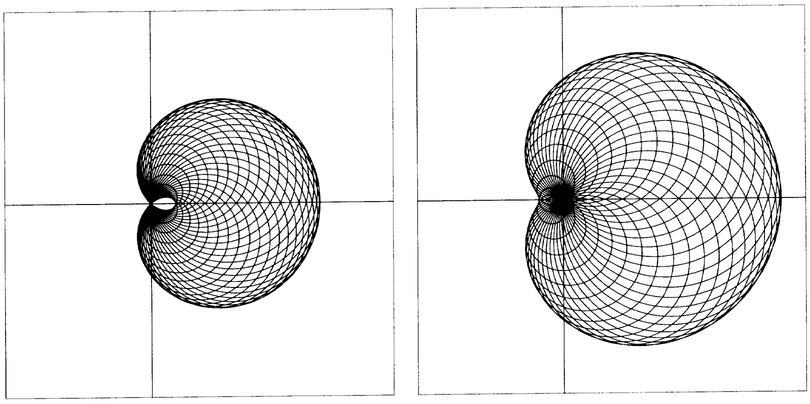

Figure 9. Two forms of the limaçon of Pascal as the boundary of a Minkowski product of two circles, when one of the circles passes through the origin.

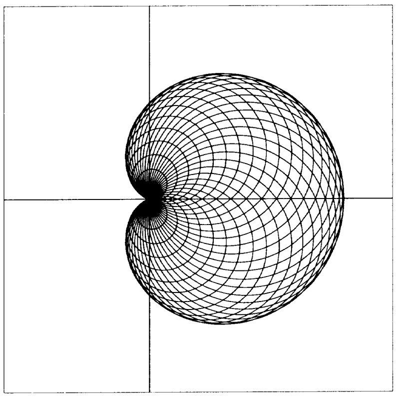

Figure 10. The cardioid as the boundary of a Minkowski product of two circles, when both circles pass through the origin.

of the exterior/interior refractive indices). The point source of the spherical waves is situated at the origin.

We now show that this construction easily generalizes to yield anticaustics for the refraction of spherical waves by more complicated surfaces.*

PROPOSITION 6. *Let $\mathcal{A}$ be a smooth curve in the complex plane, and $\mathcal{B}$ be the circle with center 1 and radius k^{-1}, where k is the ratio of refractive indices on each side of $\mathcal{A}$. Then $\partial(\mathcal{A} \otimes \mathcal{B})$ is ordinarily (a subset of) the anticaustic for refraction of spherical waves from the origin by the interface $\mathcal{A}$.*

*It is understood that we are considering surfaces of revolution, with the point source situated on the symmetry axis, so we need only consider a plane section through this axis.

Proof. Let $\mathcal{A}$ be described by the parametric curve $\mathbf{z}(t)$. Then the optical path length ℓ from the origin to $x + \mathrm{i}\,y$, via the curve point $\mathbf{z}(t)$, is

$$\ell \;=\; p\,|\mathbf{z}(t)| \;+\; q\,|x + \mathrm{i}\,y - \mathbf{z}(t)|\,.$$

Setting $\ell = 0$ and squaring, we obtain a one-parameter family of circles

$$\phi(x, y, t) \;=\; |\mathbf{z}(t)|^2 - k^2|x + \mathrm{i}\,y - \mathbf{z}(t)|^2 \;=\; 0\,, \tag{20}$$

and the anticaustic for refraction of spherical waves from the origin by $\mathcal{A}$ is the envelope of this family. Now, for each t, $\phi(x, y, t) = 0$ describes the circle with center $\mathbf{z}(t)$ and radius $k^{-1}|\mathbf{z}(t)|$, which can be obtained by multiplying $\mathbf{z}(t)$ and the circle $\mathcal{B}$. The family of circles (20) is thus the same as $\{\mathbf{z}(t)\} \otimes \mathcal{B}$ for all t, and the union of all these circles is the Minkowski product $\mathcal{A} \otimes \mathcal{B}$. In 'simple' cases, such as products of lines and circles, the boundary $\partial(\mathcal{A} \otimes \mathcal{B})$ is identical to the envelope of the family of circles. When $\mathcal{A}$ is a general curve, however, we must allow for the possibility that: (i) portions of the envelope lie in the *interior* of the region $\mathcal{A} \otimes \mathcal{B}$; and (ii) in exceptional circumstances, portions of individual members of the family (which are not considered part of the envelope) may contribute to the boundary of $\mathcal{A} \otimes \mathcal{B}$. Thus, for a general curve $\mathcal{A}$ and circle $\mathcal{B}$, we qualify equating $\partial(\mathcal{A} \otimes \mathcal{B})$ with the anticaustic by saying 'ordinarily' and 'a subset of' in Proposition 6. □

For further details on anticaustics in geometrical optics, see [7, 14, 15].

6.7. FURTHER MINKOWSKI PRODUCTS

Many other interesting geometries can be generated as Minkowski products of 'simple' curves – in this section, we present a few illustrative examples of the products of conics (ellipses and hyperbolas) with circles.

Figure 11 shows two instances of the Minkowski product of an ellipse and a circle. Here, the ellipse is defined by the equation $(x/4)^2 + y^2 = 1$, while the circle is centered at 1 and has radius r. On the left in Figure 11, we show the one-parameter family of circles comprising this product when $r = 1$. In this case, the Minkowski-product boundary is an *oval of Cassini* [36, 37] – an algebraic curve of degree 4 that can be described by the bipolar equation

$$r_1 r_2 \;=\; k^2 \tag{21}$$

with respect to two distinct poles. For a circle of radius $r = 2$, the Minkowski product exhibits the interesting form shown that is on the right in Figure 11 (the plot has been scaled by $1/2$ in this illustration).

The next example is the Minkowski product of a hyperbola and a circle. Suppose $\mathcal{A}$ is the hyperbola defined by the equation $x^2 - y^2 = 1$, and $\mathcal{B}$ is the circle of radius r centered at 1. Figure 12 illustrates the one-parameter family of circles that comprise the Minkowski product $\mathcal{A} \otimes \mathcal{B}$, for $r = 1$ and $r = 2$. In the case $r = 1$ (shown on the

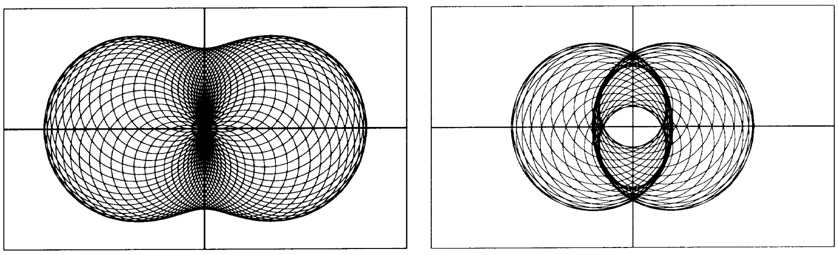

Figure 11. Minkowski product of an ellipse and circles with radii 1 and 2.

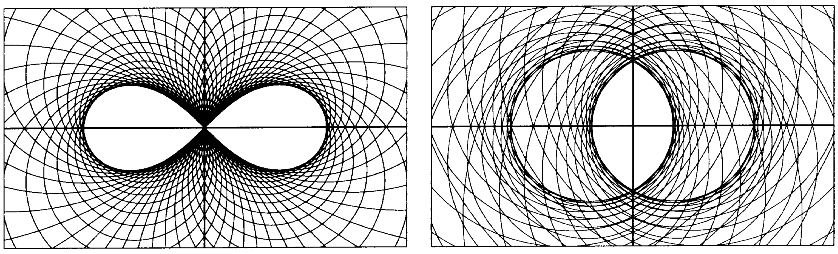

Figure 12. Minkowski product of a hyperbola and circles with radii 1 and 2

left), the boundary of the Minkowski product is a *lemniscate of Bernoulli*, which is actually a special case of the oval of Cassini, corresponding to a value for k in Equation (21) equal to half the distance between the two poles of the bipolar coordinate system.

We have focused here on Minkowski products of simple one-dimensional sets (loci). It is not difficult, however, to deduce conclusions about Minkowski products of the regions (i.e., two-dimensional sets) bounded by such loci. For example, if $\mathcal{A}$ and $\mathcal{B}$ are circular disks, one can easily see that the product $\mathcal{A} \otimes \mathcal{B}$ is the simply-connected region contained within the *outer* loop of the Cartesian oval defined by the product of the circles $\partial\mathcal{A}$ and $\partial\mathcal{B}$.

6.8. MINKOWSKI POWERS, ROOTS, AND FACTORIZATIONS

Adopting an algebraic perspective, the Minkowski product operation allows a meaningful consideration, under appropriate conditions, of the powers, roots, and factorizations of complex sets. Hence, the nth Minkowski power $\mathcal{A}^n$ of a complex set $\mathcal{A}$ is not the set of values $\mathbf{a}^n$ where $\mathbf{a} \in \mathcal{A}$, but rather the values $\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_n$ where the $\mathbf{a}_i$ are *independently chosen* from $\mathcal{A}$. Correspondingly, the n-th Minkowski root $\mathcal{A}^{1/n}$ of a set $\mathcal{A}$ is defined by the property that

$$\{\mathbf{z}_1\mathbf{z}_2 \cdots \mathbf{z}_n \mid \mathbf{z}_i \in \mathcal{A}^{1/n} \text{ for } \; i = 1, \ldots, n\} = \mathcal{A}.$$

This definition is rather indirect: it is not clear, for example, that there is a *unique* set $\mathcal{A}^{1/n}$ satisfying it. Indeed, a multiplicity of nth Minkowski roots of a given complex set $\mathcal{A}$ would not be surprising, since we know there are n distinct roots in the case that $\mathcal{A}$ is a singleton set.

In order to discuss the Minkowski factorization of a set $\mathcal{A}$, we must first specify the domain in which the factors or 'prime' sets reside. Of course, this specification will influence the factorizability of $\mathcal{A}$. We should also mention, in this context, that singleton sets amount to scalars in the geometric algebra: since any set is divisible by any singleton, except $\{0\}$, we do not count them as factors. If we take simple curves (lines and circles) as our primes, the results of Section 6 already provide several examples of such Minkowski factorizations.

The computation of Minkowski powers, roots, or factorizations of general complex sets is evidently a nontrivial task that deserves further study.

7. Minkowski Division

As noted in Section 2, the Minkowski division $\mathcal{A} \oslash \mathcal{B}$ is just the Minkowski product of $\mathcal{A}$ with the 'reciprocal set' $\mathcal{B}^{-1}$ of $\mathcal{B}$. However, division and reciprocal sets are worthy of study in their own right. For example, Möbius transformations can be built up from additions, multiplications, and reciprocations.

First, if $\mathcal{A}$ is a smooth curve C in the complex plane and $\mathcal{B}$ is the vertical line through 1, the Minkowski division $\mathcal{A} \oslash \mathcal{B}$ yields the region bounded by the pedal curve of C with respect to the origin. This can be regarded as the converse of Proposition 3, concerning the *product* $\mathcal{A} \otimes \mathcal{B}$ of the two sets; the qualifications made there also apply in the division context. Since $\mathcal{B}^{-1}$ is the circle with the interval $[0, 1]$ as diameter, $\partial(\mathcal{A} \oslash \mathcal{B})$ can be computed as the envelope of the family of circles generated by multiplying $\mathcal{B}^{-1}$ by each point of C. Introducing a parametric representation for C, and invoking the usual envelope method, one can show that the point on each circle that contributes to the envelope is the foot point from the origin to a tangent line of C.

In the context of geometrical optics, one can design the mirror that yields a given anticaustic C by taking the Minkowski product of C and the vertical line through $\frac{1}{2}$. On the other hand, given a mirror C, the Minkowski division of C by the vertical line through $\frac{1}{2}$ gives the anticaustic for reflection by C. This is an immediate consequence of Proposition 6, and the fact that the reciprocal $\mathcal{B}^{-1}$ of the vertical line through $\frac{1}{2}$ is the circle centered at 1 with radius 1 (note that the radius of this circle represents the ratio of refractive indices; a ratio of 1 corresponds to the case of reflection).

Now consider Minkowski division by a circle. Suppose $\mathcal{A}$ is a given curve C in the complex plane, and $\mathcal{B}$ is a circle normalized to have center at 1. In order to compute $\mathcal{A} \oslash \mathcal{B}$, we first need to calculate the reciprocal $\mathcal{B}^{-1}$. If $\mathcal{B}$ is of radius r, the reciprocal $\mathcal{B}^{-1}$ is the circle centered at $1/(1 - r^2)$ of radius $r/|1 - r^2|$. We can then apply the normalization procedure to $\mathcal{B}^{-1}$ by taking a scalar multiplication with $1 - r^2$ to obtain the original circle $\mathcal{B}$. Hence, the Minkowski division $\mathcal{A} \oslash \mathcal{B}$ is just a scaled version of the Minkowski product $\mathcal{A} \otimes \mathcal{B}$, as follows:

$$\mathcal{A} \oslash \mathcal{B} = \left\{ \frac{1}{1-r^2} \right\} \otimes \mathcal{A} \otimes \mathcal{B} .$$

So, up to scaling, the Minkowski division $\mathcal{A} \oslash \mathcal{B}$ also generates the anticaustic for refraction by $\mathcal{A}$, with the same ratio of refractive indices.

Suppose we restrict the operands of Minkowski products and divisions to lines and circles. It is then worth investigating the behavior of the Minkowski product under the conformal map $\mathbf{z} \mapsto 1/\mathbf{z}$. Generally, the reciprocal of the Minkowski product $\mathcal{A} \otimes \mathcal{B}$ is the Minkowski product of the reciprocal sets of $\mathcal{A}$ and $\mathcal{B}$, that is,

$$(\mathcal{A} \otimes \mathcal{B})^{-1} = \mathcal{A}^{-1} \otimes \mathcal{B}^{-1} . \tag{22}$$

This relation makes it easy to compute the reciprocals of some special curves. For example, if $\mathcal{A}$ and $\mathcal{B}$ are both lines that do not pass through the origin, we can apply (22) to compute the reciprocal of the parabola $\partial(\mathcal{A} \otimes \mathcal{B})$. Since the reciprocal of a line is a circle passing through the origin, the right hand side of (22) is a cardioid, a special case of the limaçon of Pascal (see Section 6.5).

Table I lists further interesting results, which can be easily checked using Equation (22). Note that Cartesian ovals have two shape parameters, r and k, in Equation (17). As mentioned above, although the map $\mathbf{z} \mapsto 1/\mathbf{z}$ transforms a circle centered at 1 into a circle with different center, we can transform it into the original circle by simply scaling. Thus, the two Cartesian ovals in the last row of Table I are the same Cartesian oval with different scales.

Consider now the relationship between Minkowski products/divisions and the Möbius transformation $M: \mathbf{z} \to \mathbf{w}$ defined by

$$\mathbf{w} = M(\mathbf{z}) = \frac{\mathbf{az} + \mathbf{b}}{\mathbf{cz} + \mathbf{d}} , \tag{23}$$

(where $\mathbf{ad} \neq \mathbf{bc}$) – specifically, for cases with lines or circles as the operands $\mathcal{A}$ and $\mathcal{B}$. As is well-known [9, 47, 52], Möbius transformations map the set of all lines and circles in the complex plane into itself. Thus, given two Möbius transformations M and N, the Minkowski product $M(\mathcal{A}) \otimes N(\mathcal{B})$ is one of the cases discussed previously, and one can easily identify qualitative and quantitative relations between $\mathcal{A} \otimes \mathcal{B}$ and $M(\mathcal{A}) \otimes N(\mathcal{B})$.

On the other hand, it is also interesting to investigate the effect of Möbius transformations on the results of Minkowski products, rather than on their operands, i.e., to identify relationships between $M(\mathcal{A} \otimes \mathcal{B})$ and $\mathcal{A} \otimes \mathcal{B}$. Now any Möbius transform can be decomposed into the simpler steps of scalar addition, scalar multiplication, and inversion. If M is a scalar multiplication, it can be applied to just *one* of the operands – i.e.,

$$M(\mathcal{A} \otimes \mathcal{B}) = M(\mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes M(\mathcal{B}) ,$$

Table I. *Reciprocals of conics and Cartesian ovals.*

$\partial(\mathcal{A}\otimes\mathcal{B})$	$\mathcal{A}$	$\mathcal{B}$	$\partial(\mathcal{A}^{-1}\otimes\mathcal{B}^{-1})$
parabola	line $\not\ni 0$	line $\not\ni 0$	cardioid
ellipse	circle, $r > 1$	line $\not\ni 0$	limaçon with acnode
hyperbola	circle, $r < 1$	line $\not\ni 0$	limaçon with crunode
Cartesian oval	circle $\not\ni 0$	circle $\not\ni 0$	Cartesian oval

whereas if M is an inversion, we must apply it to *both* operands – i.e.,

$$M(\mathcal{A}\otimes\mathcal{B}) = M(\mathcal{A})\otimes M(\mathcal{B}).$$

When M involves both scalar addition and inversion, however, the situation is more involved because the inverses of a given set and of a translated instance of that set do not have a straightforward relationship.

The results in Table I are based upon a specific location of the origin in relation to $\mathcal{A}\otimes\mathcal{B}$ – namely, a focus of the conics, and a pole of the Cartesian oval. Inversion with respect to a different point – such as the singular point $\mathbf{z} = -\mathbf{d}/\mathbf{c}$ of (23) – will distort the algebraic and geometric symmetries of $\mathcal{A}\otimes\mathcal{B}$ in mapping it to $M(\mathcal{A}\otimes\mathcal{B})$, and thus yields more complicated results. Thus, complex sets bounded by conics and Cartesian ovals (and their special instances) are not mapped into each other by (23), unless $\mathbf{d} = 0$.

8. Computational Considerations

In the preceding sections, we presented examples of Minkowski products with ‘simple’ operands (lines and circles), in which closed-form expressions for the Minkowski-product boundary can be obtained. As with Minkowski sums, however, the Minkowski products of general curved sets do not admit simple closed-form solutions: they will typically require extensive computations for their boundary evaluation – see, for example, [33].

We now make some fundamental observations concerning the evaluation of Minkowski products. Specifically, we present a geometrical condition that pairs of ‘corresponding points’ on two smooth curves must satisfy, if they are to yield boundary points in the Minkowski product of those curves.

Algorithms for computing Minkowski sums – especially those based on the envelope approach – usually rely [33] upon the following result:

PROPOSITION 7. *Let $\gamma(t)$ and $\boldsymbol{\delta}(u)$ be regular curves in the complex plane. If the pair of points $\gamma(t_0)$ and $\boldsymbol{\delta}(u_0)$ on these curves contributes to the boundary of their Minkowski sum $\gamma(t)\oplus\boldsymbol{\delta}(u)$, the two curves must have parallel tangent (or normal) vectors at these points – i.e., for some real nonzero λ we have*

$$\gamma'(t_0) = \lambda\,\boldsymbol{\delta}'(u_0).$$

We now present an analogous necessary condition that characterizes pairs of points on two curves that contribute to their Minkowski product boundary:

PROPOSITION 8. *Let $\gamma(t)$ and $\boldsymbol{\delta}(u)$ be regular curves in the complex plane. If the pair of points $\gamma(t_0)$ and $\boldsymbol{\delta}(u_0)$ on these curves contributes to the boundary of their Minkowski product $\gamma(t) \otimes \boldsymbol{\delta}(u)$, the condition*

$$\frac{\gamma'(t_0)}{\gamma(t_0)} = \lambda \, \frac{\boldsymbol{\delta}'(u_0)}{\boldsymbol{\delta}(u_0)} \tag{24}$$

must be satisfied for some real nonzero λ.

Geometrically, the condition (24) states that pairs of corresponding points on the two curves, which may contribute to the Minkowski product boundary, are identified by the fact that *the angle between the curve tangent vector and position vector* must be equal at those points. Whereas Minkowski sums are translation invariant, the location of the two operands relative to the origin of the complex plane clearly plays a key role in Minkowski products.

An intuitive means to prove the condition (24) is to introduce a mapping of the complex plane by the complex logarithm function. Roughly speaking, the logarithm transforms Minkowski products into Minkowski sums. Taking the logarithm of the Minkowski product $\gamma(t) \otimes \boldsymbol{\delta}(u)$, we have

$$\log(\gamma(t) \otimes \boldsymbol{\delta}(u)) = (\log \gamma(t)) \oplus (\log \boldsymbol{\delta}(u)) \,. \tag{25}$$

Thus, Proposition 8 is a straightforward consequence of Proposition 7, since the tangents to the curves $\log \gamma(t)$ and $\log \delta(u)$ are $\gamma'(t)/\gamma(t)$ and $\delta'(u)/\delta(u)$. However, the complex logarithm is a multi-valued function,

$$\log \mathbf{z} = \log |\mathbf{z}| + \mathrm{i}(\arg \mathbf{z} + 2\pi k) \quad \text{for } k = 0, 1, 2, \ldots ,$$

and proper interpretation of (25) requires a careful determination of which branch of the logarithm should be chosen along each of the curves.

To avoid these technical difficulties, we now provide a direct proof of (24) using the silhouette construction of envelopes.

Proof of Proposition 8. Suppose two sets $\mathcal{A}$ and $\mathcal{B}$ are defined by the trace of two curves, $\gamma(t) = x_1(t) + \mathrm{i}\, y_1(t)$ and $\delta(u) = x_2(u) + \mathrm{i}\, y_2(u)$, respectively. As indicated in Proposition 3 and Proposition 6, the boundary $\partial(\mathcal{A} \otimes \mathcal{B})$ of their Minkowski product is ordinarily (a subset of) the envelope of the one-parameter family of curves defined by $\gamma(t) \otimes \mathcal{B}$ (or, alternatively, by $\mathcal{A} \otimes \delta(u)$). Thus, we shall identify the condition for points $\gamma(t_0)$ and $\delta(u_0)$ to contribute to the envelope of this family of curves.

Consider the three-dimensional surface $\mathbf{r}(t,u)$ defined by

$$\mathbf{r}(t,u) = \left\{ \gamma(t)\,\boldsymbol{\delta}(u) \,,\; t \right\} \in \mathbb{C} \times \mathbb{R}$$

or, equivalently,

$$\mathbf{r}(t,u) \,=\, \left\{\, x_1(t)x_2(u) - y_1(t)y_2(u)\,,\; x_1(t)y_2(u) + y_1(t)x_2(u)\,,\; t \,\right\} \,\in\, \mathbb{R}^3\,.$$

We can imagine this surface to be obtained by 'stacking' each member t of the curve family $\gamma(t) \otimes \mathcal{B}$ at a height $z = t$ along the z-axis. The envelope of the family is then the projection of the *silhouette curve* of the surface $\mathbf{r}(t,u)$, viewed along the z-axis, onto the (x,y) plane.

The condition for a point of $\mathbf{r}(t,u)$ to lie on the silhouette curve is that the surface normal vector be perpendicular to the z direction at that point. Since the partial derivatives of $\mathbf{r}(t,u)$ are given* by

$$\mathbf{r}_t = \left\{\, x_1'x_2 - y_1'y_2\,,\; x_1'y_2 + y_1'x_2\,,\; 1 \,\right\},$$
$$\mathbf{r}_u = \left\{\, x_1x_2' - y_1y_2'\,,\; x_1y_2' + y_1x_2'\,,\; 0 \,\right\},$$

the surface normal has the direction of $\mathbf{r}_t \times \mathbf{r}_u$, namely

$$\begin{aligned}\mathbf{r}_t \times \mathbf{r}_u = \big\{ & -x_1y_2' - y_1x_2'\,,\; x_1x_2' - y_1y_2'\,,\\ & (x_1'x_2 - y_1'y_2)(x_1y_2' + y_1x_2') - (x_1'y_2 + y_1'x_2)(x_1x_2' - y_1y_2') \big\}\,.\end{aligned}$$

Now if $\mathbf{r}_t \times \mathbf{r}_u$ is orthogonal to the z direction, its z component must vanish. By expanding and rearranging, this gives the condition

$$(x_1x_1' + y_1y_1')(x_2y_2' - x_2'y_2) - (x_2x_2' + y_2y_2')(x_1y_1' - x_1'y_1) \,=\, 0\,,$$

which implies that

$$x_1x_1' + y_1y_1' : x_1y_1' - x_1'y_1 \,=\, x_2x_2' + y_2y_2' : x_2y_2' - x_2'y_2\,.$$

Hence, there is a nonzero real number μ such that

$$(x_1' + \mathrm{i}\,y_1')(x_1 - \mathrm{i}\,y_1) \,=\, \mu(x_2' + \mathrm{i}\,y_2')(x_2 - \mathrm{i}\,y_2)\,,$$

and by choosing $\mu = \lambda(x_1^2 + y_1^2)/(x_2^2 + y_2^2)$, we obtain the condition (24). □

Based on Proposition 8, an algorithm for computing the boundary of the Minkowski product of two curves can be developed by fairly straightforward modifications of the Minkowski sum algorithm described in [33]. Basically, we step along the curve $\gamma(t)$, and use condition (24) to identify corresponding points on $\delta(u)$ that (may) contribute to the Minkowski-product boundary. A preprocessing step may be invoked to find corresponding intervals for the parameters along the two curves (in the case of Minkowski sums, this is based on analysis of the Gauss maps of the two curves; for Minkowski products, an analogous 'topological analysis' must be based on condition (24)).

The outcome of this process is, in general, a collection of (approximated) curve segments – of which some are elements of the Minkowski-product boundary, while the remainder lie in the interior of the Minkowski product. The final step is thus

*We drop the parameters (t,u) henceforth, since they can be inferred by context.

to identify and discard the latter elements, and organize the true boundary elements into nested sequences of oriented loops: this can be done in substantially the same manner as for Minkowski sums [33].

Finally, we note that the above discussion is based upon the assumption that the operands $\gamma(t)$ and $\delta(u)$ are both smooth (i.e., tangent-continuous). However, appropriate provisions can be introduced to accommodate also the case of tangent-discontinuous curves. We hope to give a detailed algorithm description, addressing all these considerations, in a forthcoming paper.

9. Closure

The geometric algebra of complex sets under the Minkowski sum and product operations is an attractive and fertile field of investigation, with wide-ranging potential applications. In this introductory study, we could only address basic foundations and preliminary results in its systematic development.

Beyond the closed-form results for 'simple' operands derived in Sections 6 and 7, a comprehensive study of Minkowski products for general operands is needed, together with efficient evaluation algorithms: the geometrical condition of Section 8 offers a promising point of departure for these purposes. Another important topic is evaluation of the 'implicit' sets described in Section 4, and the elaboration of their relationship with Minkowski combinations. Some of these issues are addressed in a companion paper [17].

Other areas that merit further investigation are the problems arising from an algebraic perspective – the Minkowski powers, roots, and factorizations that were mentioned in Section 6.8 – and the behavior of Minkowski combinations under Möbius transformations and other conformal mappings (see Section 7). We hope to address some of these issues in subsequent papers.

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