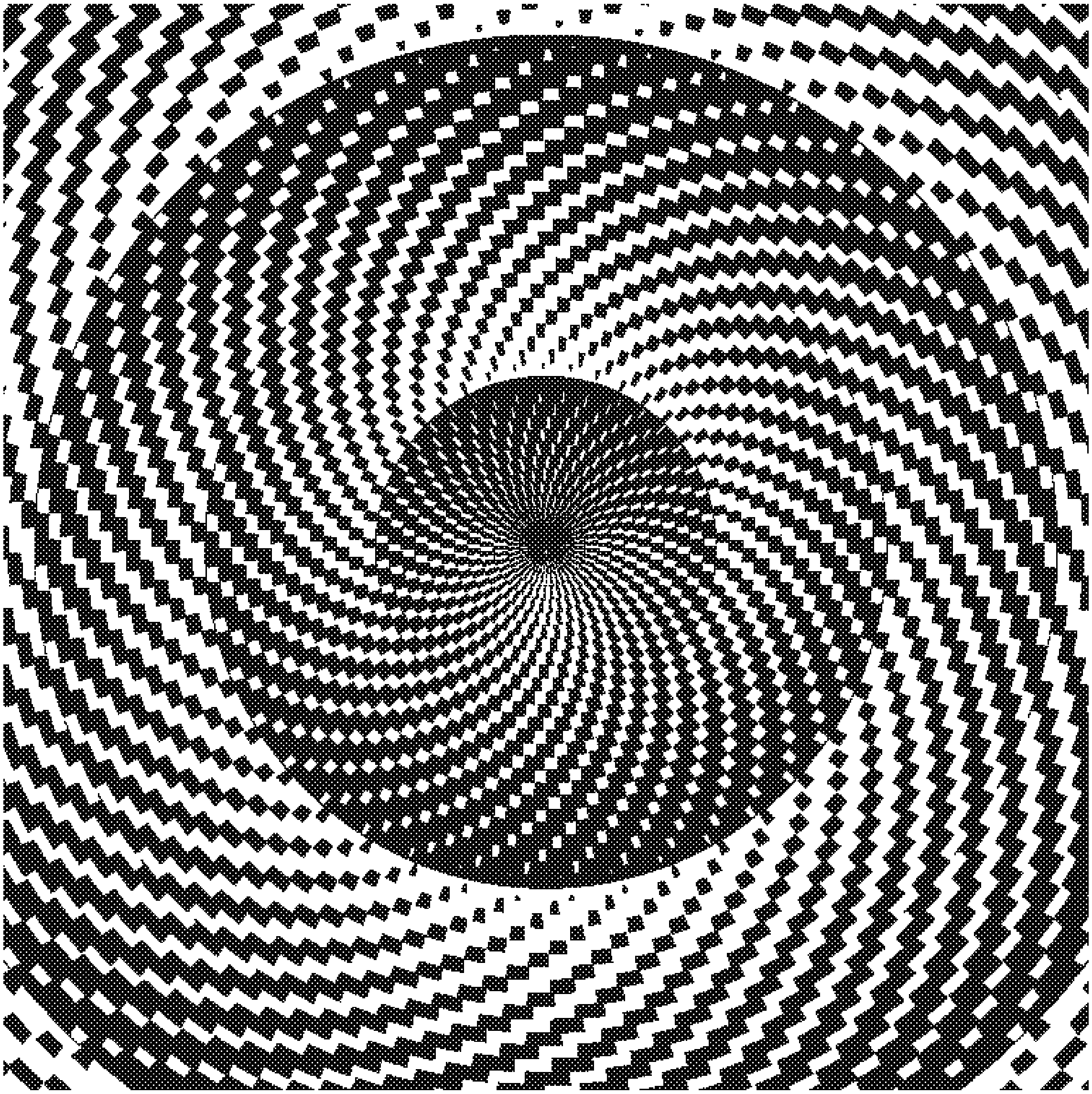


GrafEq Exercises, Examples & Enrichment Topics

$$1.9\cdot\cos\left(41\cdot T+2\cdot\left\lfloor\frac{13\cdot r}{\pi}\right\rfloor\right)<\frac{2}{\pi}\cdot\left(r-\pi\cdot\left\lfloor\frac{r}{\pi}\right\rfloor\right)+\sin\left(T+\pi\cdot\left\lfloor\frac{r}{\pi}\right\rfloor\right)-1$$
$$T=\text{angle}(x,y)$$
$$r>0$$

Created with OpenOffice.org Writer2.3 and exported to pdf format.

Introduction

GrafEq is a ‘tool’ - but a tool of perhaps somewhat more complexity than a hammer or screw driver. The novice user will assume GrafEq to be of use in graphing conics, trig and polynomial relations - the sorts of tasks addressed in a secondary mathematics curriculum. The sorts of things one might use a graphing calculator for.

However, due to the algorithm employed, GrafEq’s potential exceeds these expectations: GrafEq can be used in the study of primes and Pythagorean triplets for example. “Half-tone” plots are possible. It is hoped that schools with site licenses will make the program and these files available to students in the library.

The enclosed sample lesson sheets were designed to accompany GrafEq (Some were created for V1.x and others for V2.x). Some topics will be addressed in more than one section – for instance – once in a hand-out exercise sheet; and again as an enrichment exercise. They are available in Word™ format to site licensees in order to enable the teacher to easily modify them for use with particular hardware/network configurations or more recent versions of GrafEq. Some may be missing graphs - these can likely be created by simply following the instructions. These exercises are the property of Pedagoguery Software Inc., but may be copied for use in lessons using GrafEq. All are formatted for convenient 8.5"×11" printouts.

Note the convention used for reference to specific keys: <a> is the lowercase ‘a’; <Enter↩> is the enter key; <Ctrl> is the control key.

Comments/questions/difficulties? Contact Pedagoguery directly:

fax: (778) 373-4963

tel: (604) 629-7589

e-mail: peda@peda.com

Further information about GrafEq (and other Pedagoguery products) is available at:

http://www.peda.com

Table of Contents

Getting to the Point

(How might GrafEq be used in the Secondary Mathematics Curriculum?)

Curriculum guides often state explicitly or implicitly that the technology fills a supporting role in the pursuit of academic goals:[1] “The time saved by using calculators or computers to perform complex calculations can be used to help students better understand mathematical concepts.”...“Computers and calculators can be used as *tools* to ....” (italics added by author)

The point being that “we are teaching math - not computers.” But it may well be that the technology permits the study of topics from beyond (or maybe beside) the prescribed curriculum - including topics illustrative of atypical applications of common techniques. Such topics are practical only with the availability of the technology.

This section will provide examples of such topics. All graphing has been done using GrafEq V2.x.

The reader is encouraged to test the results on his/her computer.

★★★

Consider the exercise of determining the simultaneous solution of two equations. Usually achieved graphically by plotting the curves separately & noting intersection points. Question: Can we combine the original equations to form a single equation equivalent to the originals? Solution: consider, for instance, the line : y=x and the circle: $x^2+y^2=10^2$.

Recall the algebraic property: xy = 0 iff x=0 or y=0

Apply this property by re-expressing the equations to equal zero: y-x = 0 and $x^2+y^2-10^2=0$; then combine them to form a product:

$$(y-x)(x^2+y^2-10^2)=0 \qquad \text{A}$$

Graphing equation A does indeed ‘verify’ the validity of the technique. This method can be extended to 3 or more curves in a single equation. The alert student will realize that this exemplifies a use of the property beyond its usual “solve by factoring” function.

Now - from a slightly different perspective: we have created a single graph - the *union* of the two originals. Can we now determine the *intersection* of the two graphs by creating a single equation?

For instance, re-consider the above originals: y=x and $x^2+y^2=10^2$. We seek an equation which is true when the originals are both true - false otherwise.

$$(y-x)^2+(x^2+y^2-10^2)^2=0 \qquad \text{B}$$

This seems to work: the graph consists of the expected two dots. The general method: f=0 and g=0 are true simultaneously if $f^2+g^2=0$. Confirmation of the general validity of the technique is left as an exercise for the reader.

Both of the previous examples illustrate the difference between a technology approach: “Does it work?” and an analytic approach: “Can I prove it?”

★★★

1 **The Common Curriculum Framework for K-12 Mathematics.**: Western Canadian Protocol for Collaboration in Basic Education p11

The reflective student of secondary math might notice something rather curious: we study the equations and graphs of lines, conics, absolute value, exponential-logs, polynomials and trig – but seem to omit: points, segments, rays and squares - not to mention triangles, pentagons, hexagons, pentacles etc. With the current technology, we can let the computer do the plotting if we can come up with the defining relations:

point (4,-2): $(x\text{-}4)^2+(y—2)^2 = 0$ ... A circle with radius 0.

segment (2,3)-(1,6): $\sqrt{(x-2)^2+(y-3)^2}+\sqrt{(x-1)^2+(y-6)^2}=\sqrt{10}$ by using "between-ness", or, using domain restriction: $y=-3x+9\times\dfrac{\sqrt{x-1}\sqrt{2-x}}{\sqrt{x-1}\sqrt{2-x}}$

ray (2,3) through (1,6): $\sqrt{(x-2)^2+(y-3)^2}+\sqrt{10}=\sqrt{(x-3)^2}$

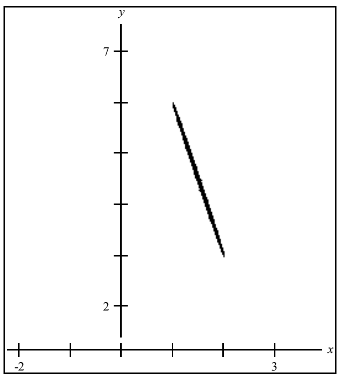


Fig.1 Segment (2,3)-(1,6)

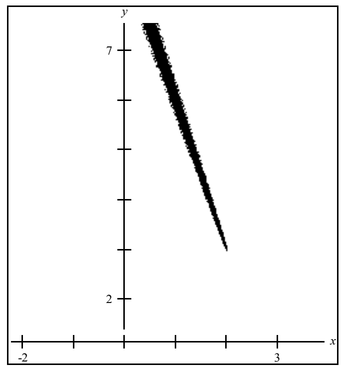


Fig.2 ray from (2,3) through (1,6)

square: 4X4 centered at (0,0): $|x|+|y|=2\sqrt{2}$

or: $\left|x\cos\frac{\pi}{4}-y\sin\frac{\pi}{4}\right|+\left|x\sin\frac{\pi}{4}+y\cos\frac{\pi}{4}\right|=2\sqrt{2}$

triangle (2,3),(1,6),(0,0) : left as exercise.

and, to change the pace - in polar - a regular pentagon: $\dfrac{r}{\sin .3\pi}=\dfrac{10}{\sin(.7\pi-\operatorname{mod}(\theta,.4\pi))}$ where $0\le\theta\le 2\pi$

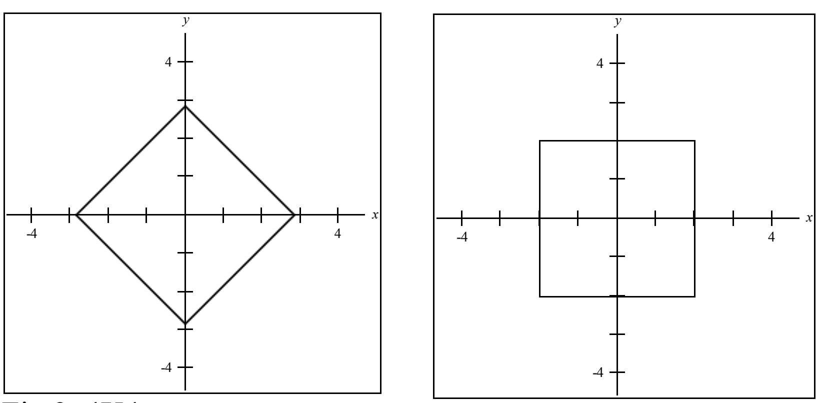


Fig.3 4X4 squares

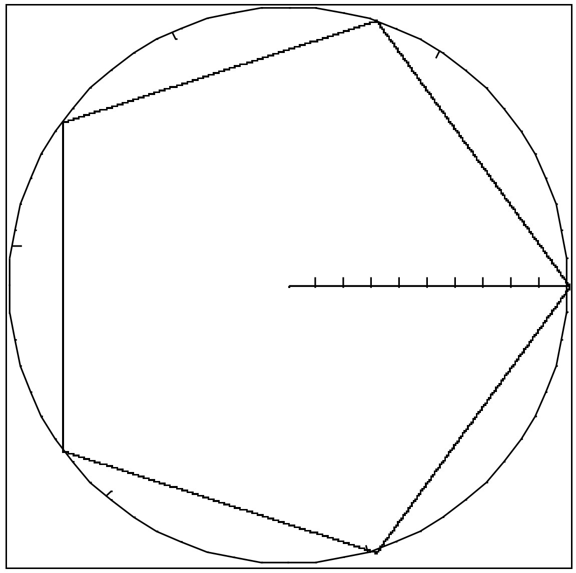


Fig.4 pentagon, plotted in a polar system

★★★

Another example of how graphing technology may enhance the curriculum: locus problems. GrafEq's multi-constraint capability is ideal for direct entry of the defining parameters of a locus exercise. Consider the following: Given two perpendicular lines, determine the locus of the point whose distance squared from one line equals twice the distance square-rooted from the other line. (With no loss of generality, we can consider the first line to be the y-axis and the second to be the x-axis.) Method: Simply enter the 3-constraint relation below:

$D^2 = 2\sqrt{d}$

$D = x$

$d = y$

The graph is shown in Fig.5. below.

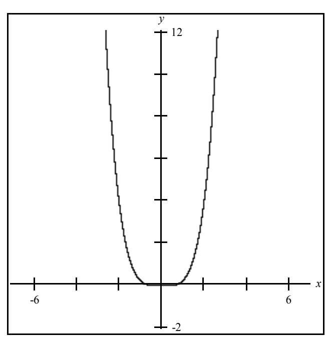


Fig.5 locus of a point (seems to be a quartic)

A final application: recall those transformation exercises wherein the student is given a piece-wise graph labeled "y=f(x)" and is asked to, say, produce the graph of "x=|f|y||". An example, using GrafEq, is illustrated below:

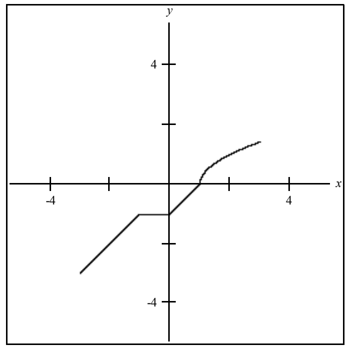


Fig.6 piece-wise graph y=f(x) as defined by

$$Y=F$$

$$F=\begin{cases} X & \text{if } -3<X<-1 \\ -1 & \text{if } -1\le X<0 \\ X-1 & \text{if } 0\le X<1 \\ \sqrt{X-1} & \text{if } 1\le X<3 \end{cases}$$

$$y=Y$$

$$x=X$$

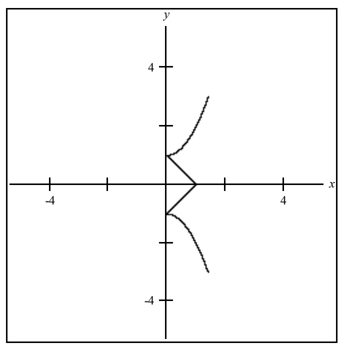


Fig.7 transformed piece-wise x=|f|y|| as defined by:

$$Y=|F|$$

$$F=\begin{cases} X & \text{if } -3<X<-1 \\ -1 & \text{if } -1\le X<0 \\ X-1 & \text{if } 0\le X<1 \\ \sqrt{X-1} & \text{if } 1\le X<3 \end{cases}$$

$$x=Y$$

$$|y|=X$$

Note the use of ‘place-holding’ variables - very handy for copy and paste operations to minimize typing.

The topics touched on in this paper are only a few of those mathematical constructs capable of pursuit through modern graphing technology. The reader is encouraged to experiment further.

Simultaneous Systems

One Equation – Many Curves

Simultaneous systems are usually visualized as the intersection of two curves determined by two equations. Two curves can, however, be defined with a single equation. This is how it can be done:

Use the algebraic property: ab = 0 if and only if a = 0 or b = 0. To graph, say, the ellipse $x^2/4+y^2/8 = 1$ and the parabola $x = y^2+2y-3$, re-express each equation so that the right side is 0:

$x^2/4+y^2/8-1 = 0$ and $y^2+2y-3-x = 0$. Then create the single equation by multiplying the modified equations and set their product to 0. You now have the single equation: $(x^2/4+y^2/8-1)(y^2+2y-3-x) = 0$ which will produce a 'parabellipse' when graphed.

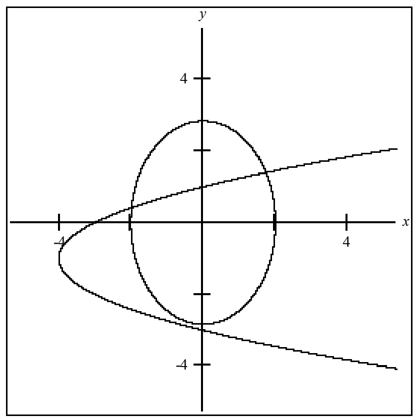
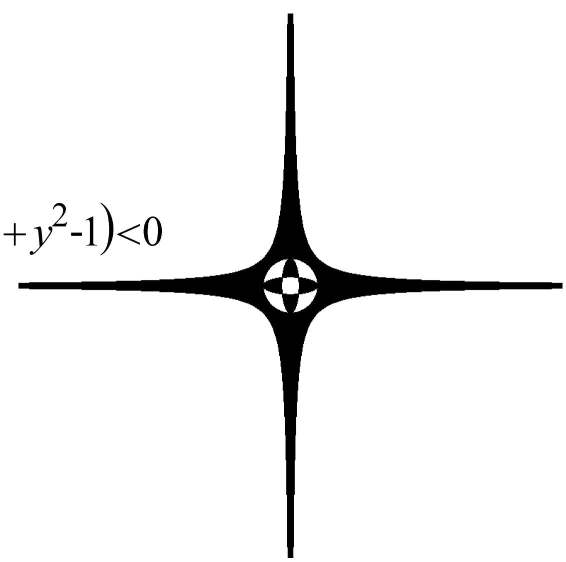


Fig.8 $(\frac{x^2}{4}+\frac{y^2}{8}-1)(y^2+2y-3-x)=0$

There are obviously two simultaneous solutions: in Q1 and Q2 and a near miss in Q3.

This method may be extended to plot more than 2 curves via a single relation.

$$(x\cdot y-1)\cdot(x\cdot y+1)\cdot\left(x^2+y^2-1\right)\cdot\left(x^2+10\cdot y^2-1\right)\cdot\left(10\cdot x^2+y^2-1\right)<0$$



The Difference Equation

Student Exercise - Enrichment

A challenging problem for the capable student - perhaps most appropriately posed after study of the concept of “composite equations ”

“Can two equations be solved by graphing a single equation?”

Consider the following example :

I $y = x^2+2x-1$
II $x^2+y^2 = 9$

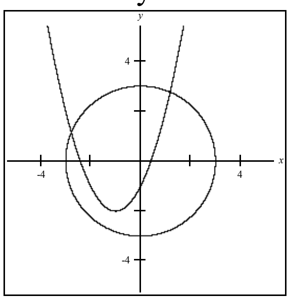


Fig.9

The graphs of the above pair is displayed in Fig.9.

Note that there is a simultaneous solution near (-3,1).
This is the method to replace I and II with a single equation: first re-express II explicitly for y:

II': $y = \pm\sqrt{-x^2+9}$

Consider a third equation (II' - I): III: $y = \pm\sqrt{-x^2+9} - x^2 - 2x + 1$ the difference equation.

The graph of III is displayed in Fig.10. After two successive zoom ins on the leftmost x-intercept, an x-value of -2.773 will be determined (via the 1 point View window mode) Upon substituting this value for x in equation I, a y-value of 1.144 will be ascertained. So, the simultaneous solution is near (-2.773, 1.144). How accurate are these values? Let us substitute them in the left hand side of II: we determine the sum of -2.7732 + 1.1442 to be 8.998 approximately. In Fig.9 we see another solution is near (1,3). Enter equation III in GrafEq and confirm that the rightmost x-intercept similarly determines an accurate location of the rightmost simultaneous solution.

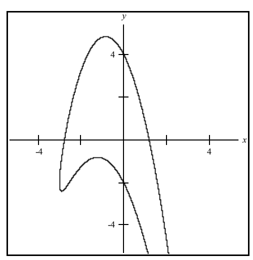


Fig.10 $y = \pm\sqrt{-x^2+9} - x^2 - 2x + 1$

Can you determine the method to determine the y-value of the solution graphically without resorting to substitution in the original equation?

Exercise: Once you feel you know the method of combining the two original equations, confirm your hypothesis by determining the simultaneous solution of I: $3x^2 + 2y^2 = 4$ and II: $y^2-x^2 = 1$ which is near (.5,1). Check your answer by substitution in the original equations.

Simultaneous Systems

Teacher Demo

(To be used with an overhead projection panel, large screen monitor or lan screen sharer)

I: $x*y = 6$ II: $x^2+y^2 = 13$

EXERCISE: Determine the simultaneous solution(s) of the above system of equations (graphically).

1. Launch GrafEq by double-clicking on its icon. Click (twice) to dispose of the title screen.
2. Enter equation I in the first relation window and graph it over the default viewport bounds (±10).
3. Under File, select New Relation, and enter equation II. Activate the second relation by <Enter↩>.

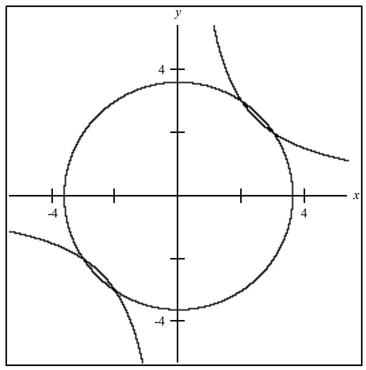


Fig.11

There appear to be four points of intersection.

4. Enter <z> to implement zoom mode. Turn 'keep this view after zooming' ON. Carefully place the zoom frame over an intersection point. Reduce the size of the box (in order to maximize magnification) with the <←> and <↓> keys. Once the small zoom frame is placed accurately over an intersection point, click the mouse button.

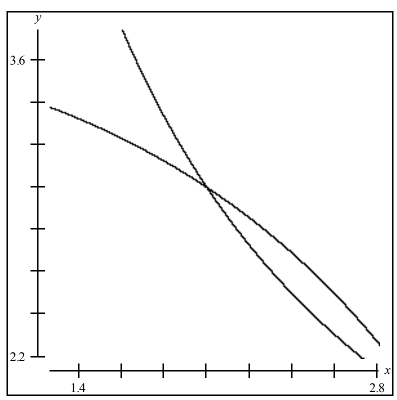


Fig.12

Fig. 12 illustrates the resultant viewport (highest simultaneous intersection point.)

5. Enter <1> to implement the one-point mode. Carefully locate the mouse over the intersection point as follows: first position the pointer with the mouse, then use the arrow keys and magnification port for precision. The above solution is near (2.007 ± .0015, 2.9995 ± .0015) which rounds to (2,3). Since the previous window (unzoomed) still exists, you can close the current zoom window to reveal the initial view window. The three remaining solutions may be similarly determined by repeating steps 4 and 5.

Slope-intercept Form

Slope-Intercept Form (y-intercept)

Student Assignment

y = mx + b

1.Launch GrafEq by double-clicking its icon. Click (twice) to dispose of the title screen.

2. In the relation window, enter the following:

y=mx+b
m=1
b=0

Use <Tab> to proceed from constraint to constraint and terminate the relation with <Enter↩>

3. The default bounds (±10) are acceptable, so click on *Create* in the Create View window.

4. Copy the graph onto the coordinate system below. Label it 'y = 1x + 0'. Make a large dot where the line crosses the y-axis and label the dot 0. This point is called the y-intercept and its y-value is 0.

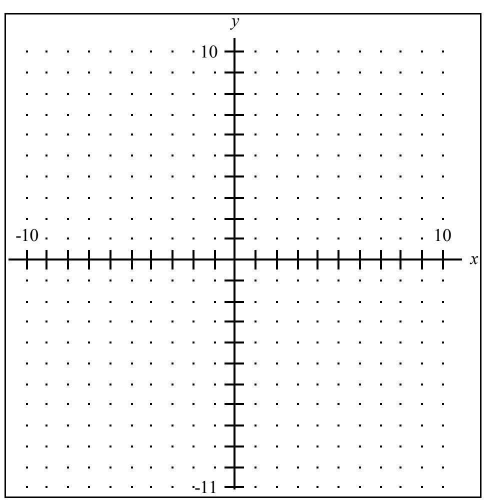
5. Activate the relation window (by clicking on it) and click on the third constraint field ('b=0') in order to edit it to read: 'b=-3', and re-activate by either entering <Enter↩> or by clicking on the relation window's active box.

6. Add this graph onto the coordinate system below and label it with its equation (y=x-3). What is the y-intercept of this line? ______

Label the dot where the line crosses the y-axis with a '-3'.

7. Activate the relation window (by clicking on it) and click on the third constraint field ('b=-3') in order to edit it to read: 'b=5', and re-activate by either entering <Enter↩> or by clicking on the relation window's active box.

8. Copy this graph to the coordinate system and label it with its equation. What is the y-intercept of this line? _______ Label the intercept on the coordinate system.



9. Where will the line with equation y = x + 7 cross the y-axis? _____ Edit b to 7 in order to confirm your answer by graphing y = x + 7.

10. Where do you think the graph of y = 2x + 5 will cross the y-axis? _________

11. Reactivate the relation window if it is not already 'up front' and edit the second constraint to read: m=2 and the third constraint to read: b=5. Activate the graph and copy it onto the coordinate system below.

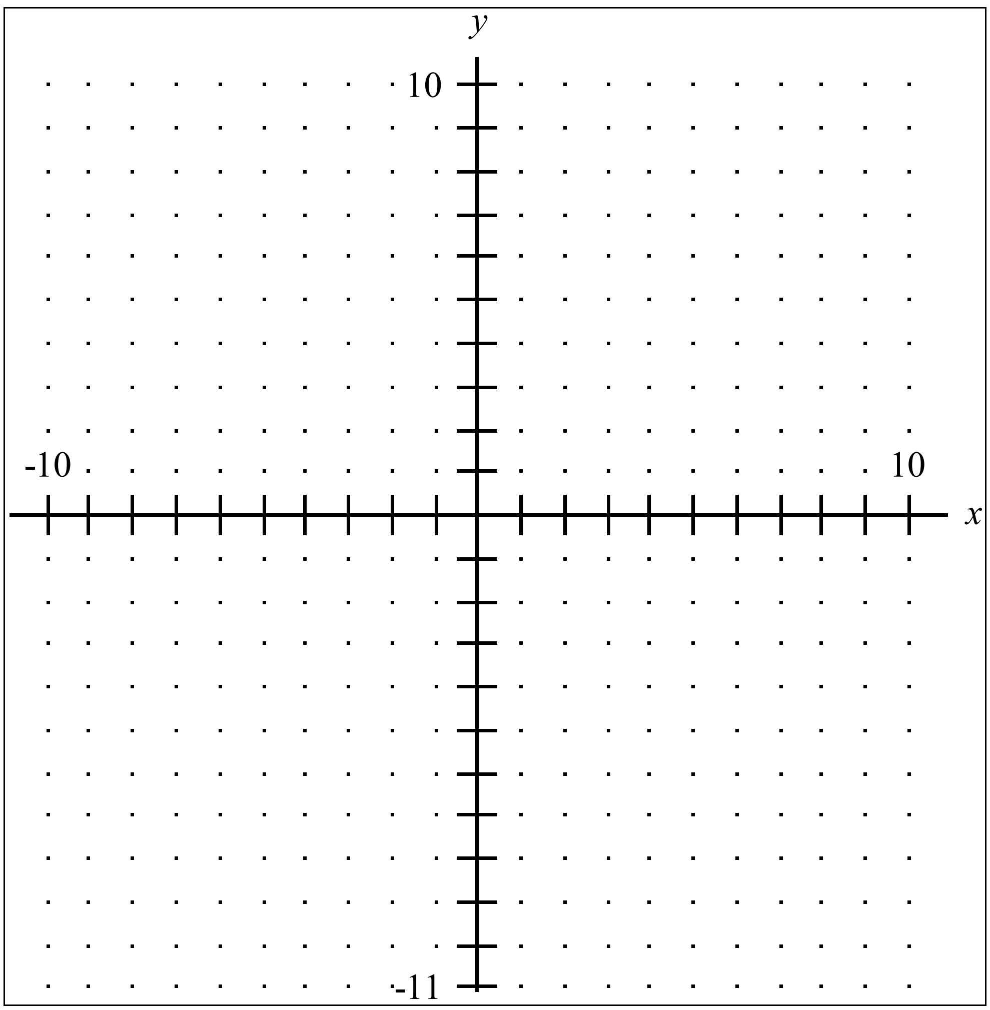
12. On the same viewport, produce the graphs of : y = -4x + 5, y = .2x + 5 and y = 3x + 5 by successively editing the values for m.

13. Copy their graphs to the system below. What do the three lines have in common?______ Why?

____

CONCLUSION:

The graph of y = mx + b is a line with y-intercept = __



Slope-Intercept Form (slope)

Student Assignment

y = mx + b

1.Launch GrafEq by double-clicking its icon. Click (twice) to dispose of the title screen.

2. In the relation window, enter the following:

y=mx+b
m=1
b=0

Use <Tab> to proceed from constraint to constraint and terminate the relation with <Enter↩>

3. The default bounds (±10) are acceptable, so click on *Create* in the Create View window.

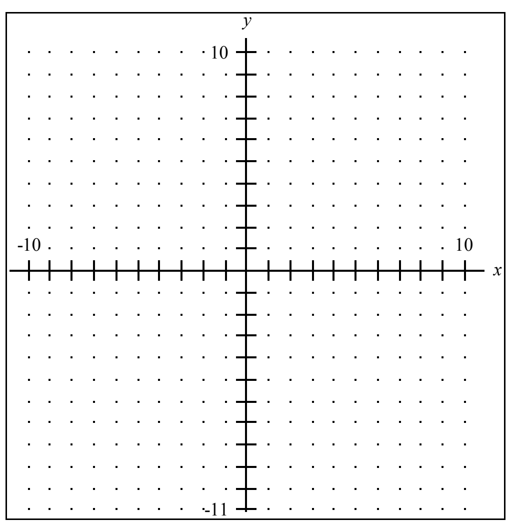
4. Copy the graph onto the coordinate system below. Label it "y=1∗x+0". The line has a slope of 1. Label the line with "m=1".

5. Activate the relation window (by clicking on it) and click on the second constraint field ('m=1') in order to edit it to read: 'm=-3', and re-activate by either entering <Enter↩> or by clicking on the relation window's active box.

6. Add this graph onto the coordinate system below and label it with its equation ('y=-3x+0'). What is the slope of this line? ______ Label the line with 'm=-3 '.

7. Activate the relation window (by clicking on it) and click on the second constraint field ('m=-3') in order to edit it to read: 'm=5', and re-activate by either entering <Enter↩> or by clicking on the relation window's active box.

8. Copy this graph to the coordinate system and label it with its equation. What is the slope of this line? _______ Label the slope on the coordinate system.



9. What will be the slope of the line with equation y = 7x + 0 ? _________ Edit m to 7 in order to confirm your answer by graphing y = 7x + 0.

10. What do you think the graph of y = 2x + 5 will have as slope? _________

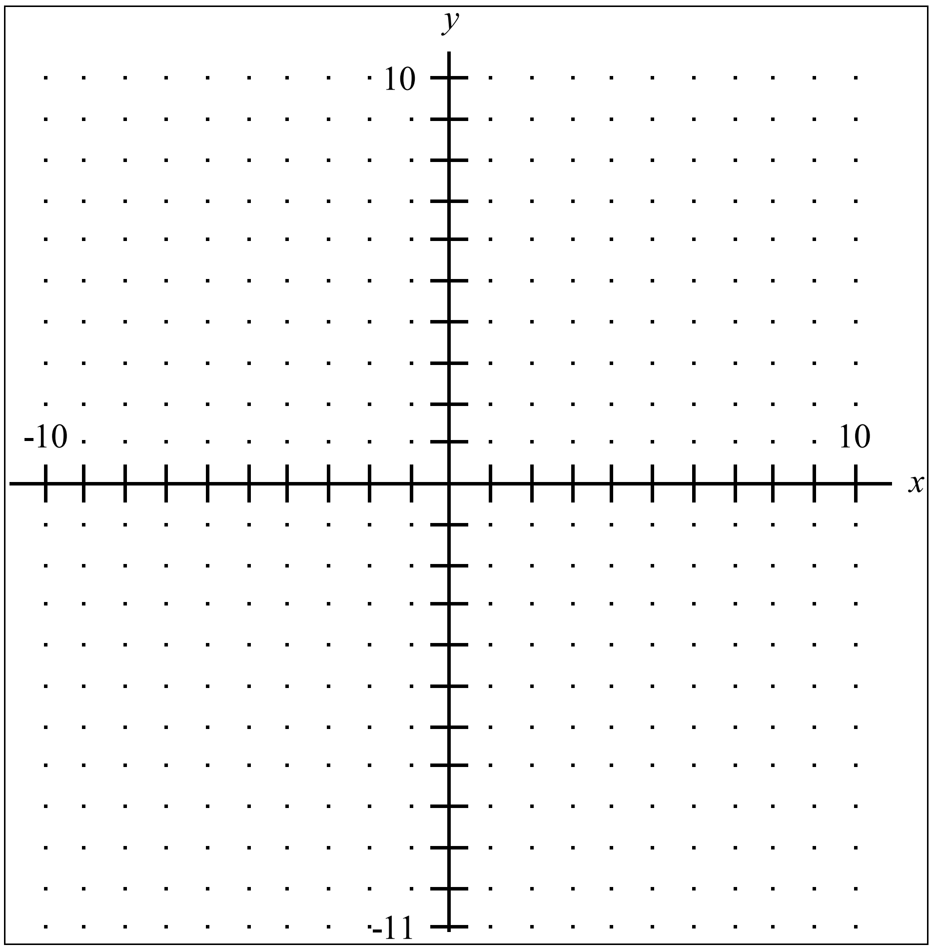
11. Reactivate the relation window if it is not already 'up front' and edit the second constraint to read: m=2 and the third constraint to read: b=5. Activate the graph and copy it onto the coordinate system below.

12. On the same viewport, produce the graphs of : y = -4x + 5, y = .2x + 5 and y = 3x + 5 by successively editing the values for m.

13. Copy their graphs to the system below. What do the three lines have in common?____________ Why?____

CONCLUSION:

The graph of y = mx + b is a line with slope = _______



Transformations

The Dilation Transformation

Student exercise

The dilation transformation is effected by 'stretching' or 'shrinking' a curve. We can do this algebraically by replacing x with x/a (or y with y/b).

It is assumed that students are knowledgeable about the graph of the parabola: $y=x^2$.

1. Launch GrafEq by double-clicking on its icon. Click (twice) to dispose of the title screen.

2. Enter the equation: $y=x^2$ by these keys: y = x↑2. (or – y = x ^ 2)

3. In the Create View window, set the y-bounds to -3 and 10 and the x-bounds to -7 and 7. Click on the Create button. If there are no visible axes enter <t>to access the ticks mode and turn ticks ON (i.e. active). The graph will appear as in Fig.13.

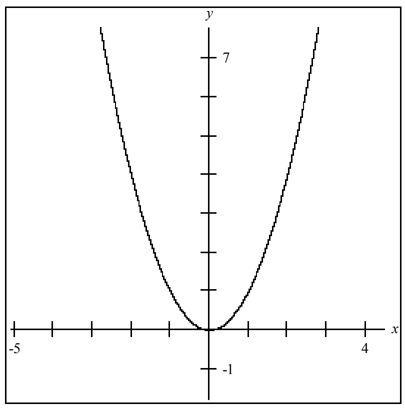


Fig.13 $y=x^2$

This graph is our basic curve - and we shall compare future graphs to it. Copy this graph onto the graph paper below (Fig.14), and label the graph '$y=x^2$. We shall therefore create a second relation to plot other graphs.

4. Under Graph, select New Relation.

5. Enter $y=(x/3)^2$ in the relation #2 window and enter <Enter↩>

Copy the graph to the graph paper and label it '$y=(x/3)^2$'.

Note that the graph of $y=(x/3)^2$ is not the same shape and size as the graph of $y=x^2$ but has been dilated by a factor of 3 in the x-(horizontal) direction.

6. Activate the relation#2 window (by either clicking on it or selecting relation#2 under Graph). Click on the edit/entry field and edit relation#2 to read: "$y=(2x)^2$".

Press <Enter↩> to activate the relation and examine the graph produced. Carefully copy it to the graph paper containing the basic curve ($y=x^2$) and the graph from 5 above. How is the graph of $y=(2x)^2$ derived from the graph of the basic ($y=x^2$) curve? ____________________________________

7. Reactivate the relation# 2 window and edit the relation to read: $y/2 = x^2$. How does the graph produced compare to the basic ($y=x^2$) curve? ____

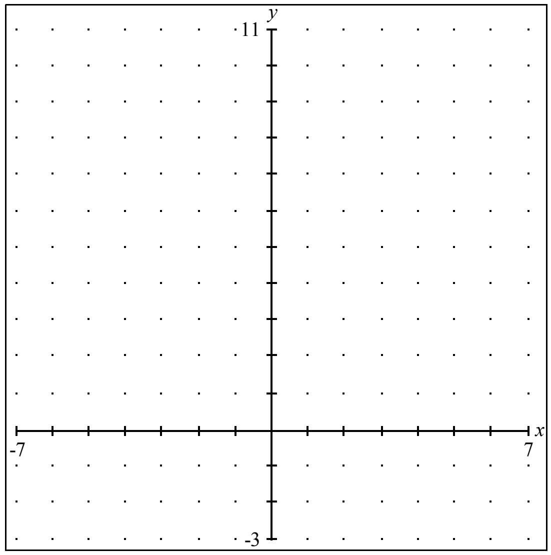


Fig. 14

8. SUMMARY: If x is replaced by x/a then the graph ____

If y is replaced with y/b then the graph ____

9. Confirmation. The graph of $x^2+y^2=16$ is shown below. What will the graph of $(x/2)^2+(3y)^2 = 16$ be? Sketch your answer onto the coordinate system. Confirm your answer by using GrafEq.

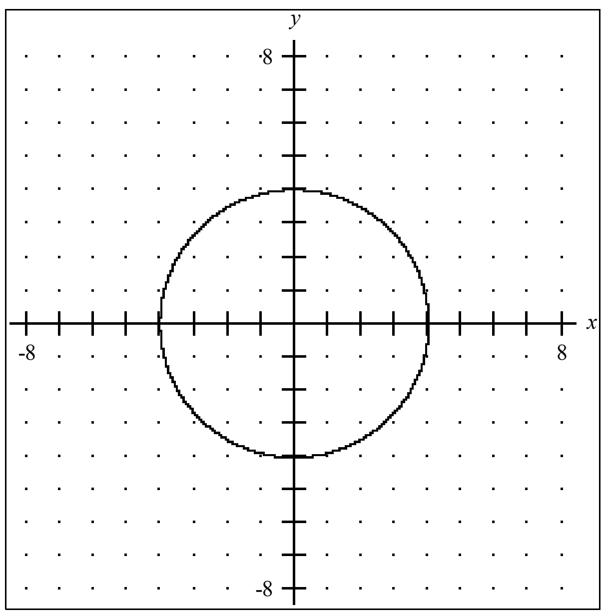


Fig.15

The Inverse Relation

Student exercise

What happens to the graph of a relation if we interchange x and y?

It is assumed that students are knowledgeable about the graph of the quadratic function : $y=x^2-3$.

1. Launch GrafEq by double-clicking on its icon. Click (twice) to dispose of the title screen.

2. Enter the equation: $y=x^2-3$ by these keys: y = x ↑ 2 - 3.

3. In the Create View window, accept the default x- and y- bounds (±10). Click on the Create button. If there are no visible axes enter <t>to access the ticks mode and turn ticks ON (i.e. active). The graph will appear as in Fig. 16.

This graph is our basic curve - and we shall compare future graphs to it. Copy this graph onto the graph paper below (Fig. 17), and label the graph '$y=x^2-3$'. We shall now create a second relation to plot another graph.

4. Under Graph, select New Relation.

5. Enter $x=y^2 - 3$ in the relation #2 window and enter <Enter↩>. The relation derived from the original by interchanging variables is called the 'inverse'.

Copy the graph to the graph paper and label it '$x=y^2-3$'.

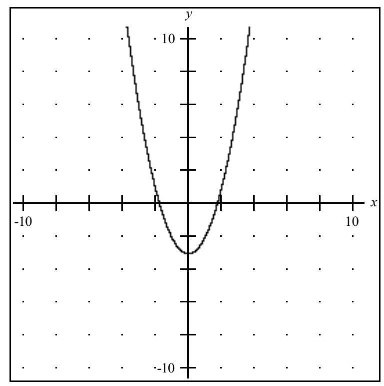


Fig.16 $y = x^2 - 3$

Note that the graphs are the same shape and size but the original has been 'flipped' over the line: x=y when inversed.

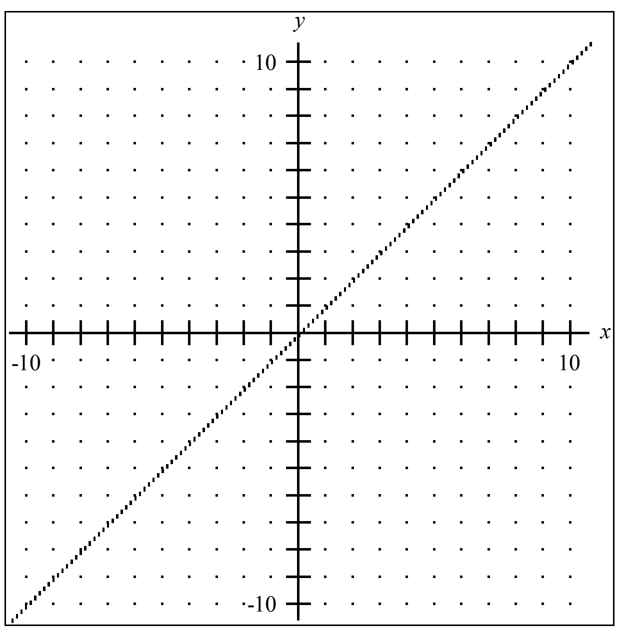


Fig.17

6.Some observations: the original graph is a 'vertical' parabola; its inverse is ___________________.

The original had x-intercepts: ±1.7; the inverse has ___________________________ of ±1.7.

The original had a y-intercept of -3; the inverse has __________________________________.

The original intersected the line y=x at the same point as ______________________________.

7. SUMMARY: A relation and its inverse:

are the same size and shape (ie 'congruent')

are symmetric over the line: y=x

will intercept each other only on the line: _____________, and

the x-intercepts of one are the ____

and vice-versa

where one relation is very 'vertical', its inverse will be very __________________.

8. Exercises: Use GrafEq to plot the following relations. Copy each graph to graph paper - one graph per coordinate system. Ensure that ticks are ON to assist in accurate copying from computer monitor to graph paper. Then draw each relation's inverse freehand - without using GrafEq.- on the same coordinate system as the original. Then check your inverse graphs with GrafEq.

relations:

a) $y=x^3$

b) $y=|x-2|+1$

c) $x^2+(y-1)^2=64$

d) $\frac{x^2}{16}+\frac{y^2}{64}=1$

9. What can you conclude about inverses with respect to 'functionality'? i.e. if a relation is a function, is its inverse also? ______________ A curve is a function if it passes the 'vertical-line test'. Can you conceive of a test for a relation to determine if its inverse is a function?

____

Polar Transformations

With access to GrafEq, secondary students are easily able to explore and extend their knowledge beyond the limits of curricular requirements.

An example is the topic of transformations - as they apply to graphs displayed in polar coordinates. Let us examine the effect upon a graph when we replace θ with (θ-k).

Note: the following equations are very slowly plotted on the earlier computers. If you have access to GrafEq V1.xx , that program can accomplish polar equations much faster.

$$r = 8\cos 3\theta$$

$$-10<r<10$$

$$-2\pi<\theta<2\pi$$

As an original curve, we shall plot the 3-leaved rose defined by $r=8\cos 3\theta$ (see Fig.18). Now if we replace θ with $(\theta-\pi/18)$ and superimpose the graph of the resultant equation: $r = 8\cos(3(\theta-\pi/18))$ we see the graphs displayed in figure 19.

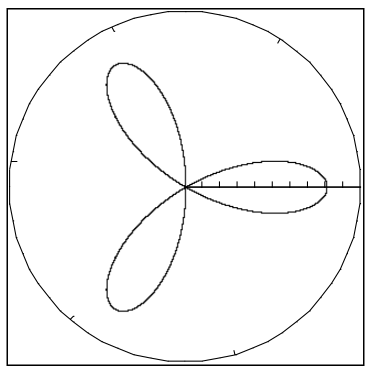


Fig.18 **I**: $r = 8 \cos 3\theta$

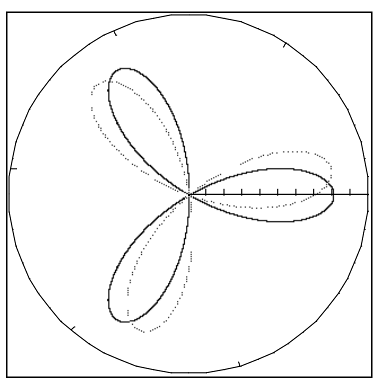


Fig.19 **I** and **II**: $r = 8 \cos 3(\theta - \pi/18)$

We appear to have 'rotated' the original curve π/18 radians in the counter-clockwise (positive) direction. If we were to graph $r = 8\cos 3(\theta - \pi/18)$ we would see the graph rotated in a clockwise direction. We can generalize to state that replacing θ with (θ-k) will effect a rotation of the curve about the origin (clockwise if k is negative; counterclockwise if k is positive). We further note that this rotation transformation is far more convenient in the polar situation than it is in rectangular Cartesian coordinates.[2] Let us now investigate the effect upon the graph by replacing r with r-2. The original graph (r = 8cos3 θ in bold) and the graph of r-2 = 8cos3 θ are shown below in fig. 20.

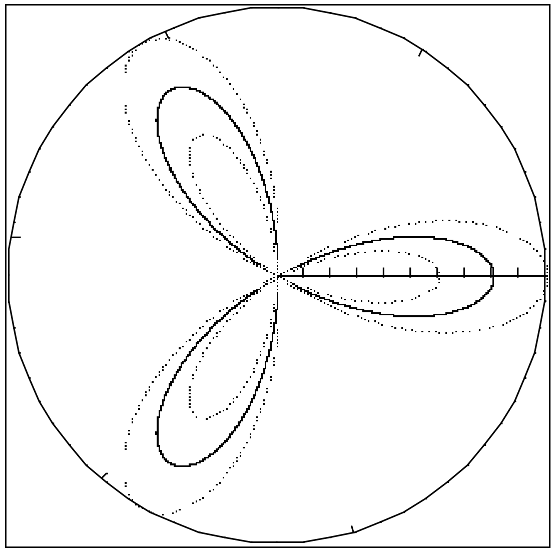


Fig.20 **I** and **III**: r-2 = 8cos3θ

The resultant graph **III** appears to have "double" petals! How can this be? If we take a close look at the original equation and its graph we can explain the double petal phenomenon.

Let us examine how **I** is plotted:

> As θ varies from 0 to π/6, cos3θ will range from 1 to 0 and r will range from 8 to 0 which is the half-petal in quadrant 1. As θ varies from π/6 to π/2, cos3θ will range from 0 to -1 to 0 and r will range from 0 to -8 to 0 (forming the petal in quadrant 3). As θ continues from π/2 to 5π/6, r will range from 0 to 8 to 0 - the petal in quadrant 2. The half-petal in quadrant 4 is plotted as θ varies from 5π/6 to π. Note that the whole rose has been plotted although θ has only ranged over one half of a full 2π rotation. Note also that one petal has positive r, one negative r and the third both. As θ proceeds to range from π to 2π the rose will be completely redrawn - and those parts that had positive r-values are now negative and vice-versa. Now we can understand the two petals in the graph of **III**: while θ has a value that determines a positive r value, the effect of the -2 is to increase the distance of the associated point by 2 (the tip of the petal will be 10 from the origin). While θ has a value that determines a negative r, the effect of the -2 will also increase the value of r - but since r is negative, the **distance** will be *decreased* by 2. (The tip of the petal will be *increased* from -8 to -6 from the origin.) It is interesting to note that with this particular case, replacing r with r+2 would have produced the same polar curve as our replacement.

The foregoing experiments have clearly shown that the effects of such simple replacements are quite different in rectangular and polar systems.

What exactly was the effect of the second replacement? Are the (three) roses all the same shape ('similar' in geometric language)? Or, in transformational terms, does the replacement effect a 'dilation'?

That question can be answered by analysis - but a more direct tack is to examine another graph - a line, say, and make the same replacement, and note the effect of that replacement.

2 To rotate a graph in rectangular coordinates, we replace x with xcosß - ysinß and replace y with xsinß + ycosß, where ß is the rotation in radians.

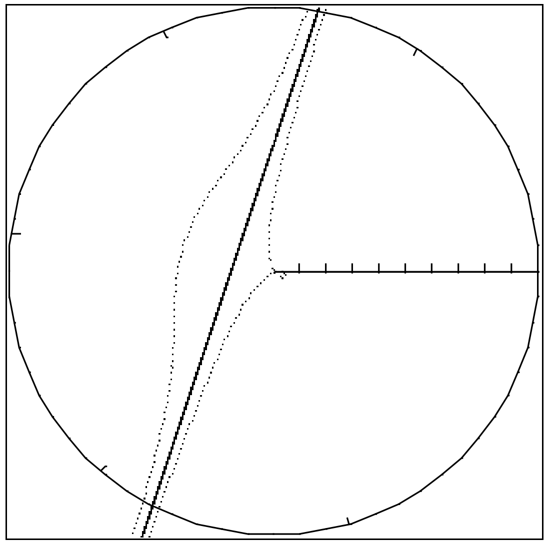


Fig.21 **IV**: rsin θ = 3rcos θ + 5 and **V:** (r-2)sin θ = 3(r-2)cos θ + 5

Fig.21 above illustrates the graph of rsin θ = 3rcos θ + 5 (the bold line) and the graph of (r-2)sin θ = 3(r-2)cos θ + 5 (the lighter lines). This example clearly illustrates that the effect of replacing r with (r-k) has no simple equivalent in the realm of rectangular transformations and in general will cause a change in shape.

The foregoing examples are instances where the computer can pique students' interest, perform the tedious calculations and visually display unexpected results. Nevertheless, to actually understand what the technology displays, the student will still need to apply traditional analytic methods.

further exercises:

1. Explore the implications of replacing: r with r/k; θ with θ/k; r with -r , θ with -θ; and interchanging r with θ.

2. Describe the effect illustrated in Fig.21 above. Hint: construct and examine a segment connecting the three curves to the origin.

The Reflection Transformation

Student exercise

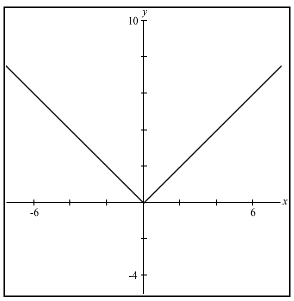
What happens to the graph of a relation if we replace x with -x (or replace y with -y)?

It is assumed that students are knowledgeable about the graph of the absolute value function : y=|x|.

1. Launch GrafEq by double-clicking on its icon. Click (twice) to dispose of the title screen.

2. Enter the equation: y=|x|

3. In the Create View window, set the y-bounds to -5 and 10 and the x-bounds to -7 and 7. Click on the Create button. If there are no visible axes enter <t>to access the ticks mode and turn ticks ON (i.e. active). The graph will appear as in fig.22.



This graph is our basic curve - and we shall compare future graphs to it. Copy this graph onto the graph paper below (fig. 23), and label the graph ‘y=|x|’. We shall therefore create a second relation to plot another graph.

Fig.22 y=|x|

4. Under Graph, select New Relation.

5. Enter -y=|x| in the relation #2 window and press <Enter↩>

Copy the graph to the graph paper and label it ‘-y=|x|’.

Note that the graph of -y= |x| is the same shape and size as the graph of y=|x| *but has been ‘flipped’ over the x-axis in the y-(vertical) direction.*[3]

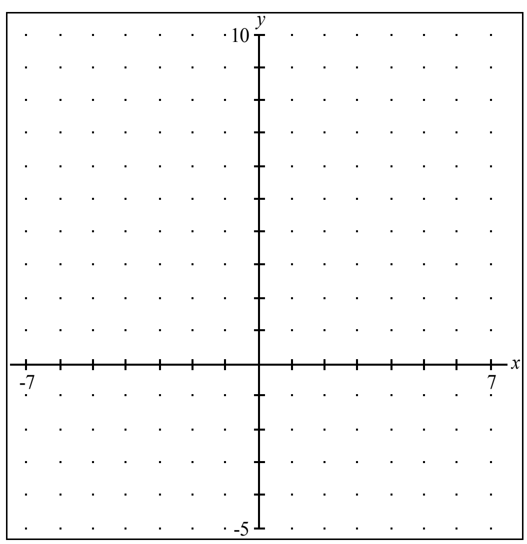


Fig 23

3 (4,4) on y=|x| becomes (4,-4) on -y=|x| when we replace 4 with its opposite.

6. Activate the relation#2 window (by either clicking on it or selecting relation#2 under Graph). Click on the edit/entry field and edit relation#2 to read: “y=|-x|”.

Press <Enter↩> to activate the relation and examine the graph produced. There appears to be no difference from the original y=|x| graph. But this is to be expected, since |x| and |-x| will be the same for all numbers.

To better illustrate the effect of replacing x with -x, try graphing $x=y^2$ and $-x=y^2$ on the same ‘New’ graph, under File. The resulting view is shown below. We note that the graph has been reflected over the y-axis.

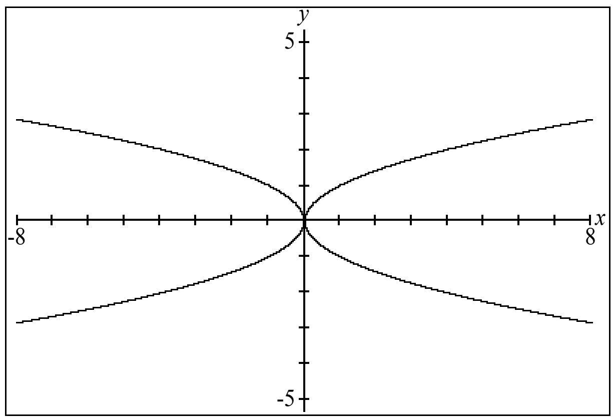


Fig.24

7. SUMMARY: If x is replaced by -x then the graph ____________________________________

If y is replaced with -y then the graph ____

The Translation Transformation

Student exercise

The simplest transformation is the "translation": replacing x with x-h (or y with y-k).

It is assumed that students are knowledgeable about the graph of the parabola: $y=x^2$.

1. Launch GrafEq by double-clicking on its icon. Click (twice) to dispose of the title screen.

2. Enter the equation: $y=x^2$.

3. In the Create View window, set the y-bounds to -3 and 7 and the x-bounds to -5 and 5. Click on the Create button. If there are no visible axes enter <t> to access the ticks mode and turn ticks ON (i.e. active). The graph will appear as in Fig.25.

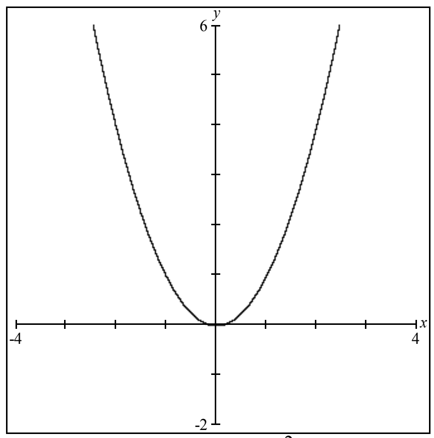


Fig.25 $y=x^2$

This graph is our basic curve - and we shall compare future graphs to it. Copy this graph onto the graph paper below (Fig.26), and label the graph '$y=x^2$'. We shall therefore create a second relation to plot other graphs.

4. Under Graph, select New Relation.

5. Enter $y=(x-3)^2$ in the relation #2 window and press <Enter↩>.

Copy the graph to the graph paper and label it '$y=(x-3)^2$'

Note that the graph of $y=(x-3)^2$ is the same shape and size as the graph of $y=x^2$ but has been translated 3 to the right.

6. Activate the relation#2 window (by either clicking on it or selecting relation#2 under Graph). Click on the edit/entry field and edit relation#2 to read: "$y=(x+2)^2$".

Press <Enter↩> to activate the relation and examine the graph produced. Carefully copy it to the graph paper containing the basic curve ($y=x^2$) and the graph from 5 above. How is the graph of $y=(x+2)^2$ derived from the graph of the basic ($y=x^2$) curve?

____

7. Reactivate the relation# 2 window and edit the relation to read: $y-2 = x^2$. How does the graph produced compare to the basic ($y=x^2$) curve?

____

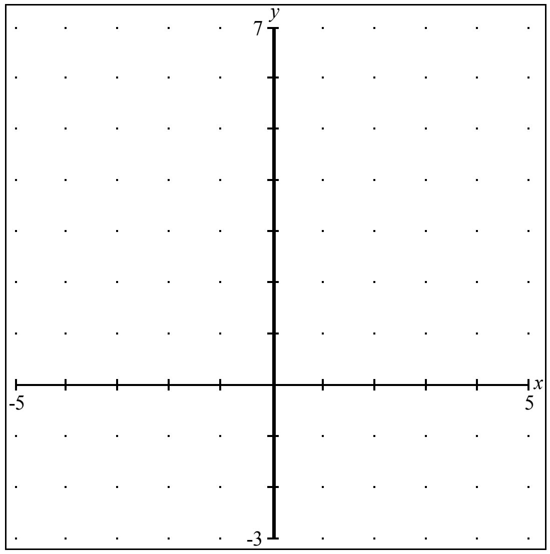


Fig.26

8. SUMMARY: If x is replaced by x-h then the graph

____

If y is replaced with y-k then the graph

____

9. Confirmation. The graph of $x^2+y^2=16$ is shown below. What will the graph of $(x-2)^2+(y+1)^2 = 16$ be? Sketch your answer onto the coordinate system. Confirm your answer by using GrafEq.

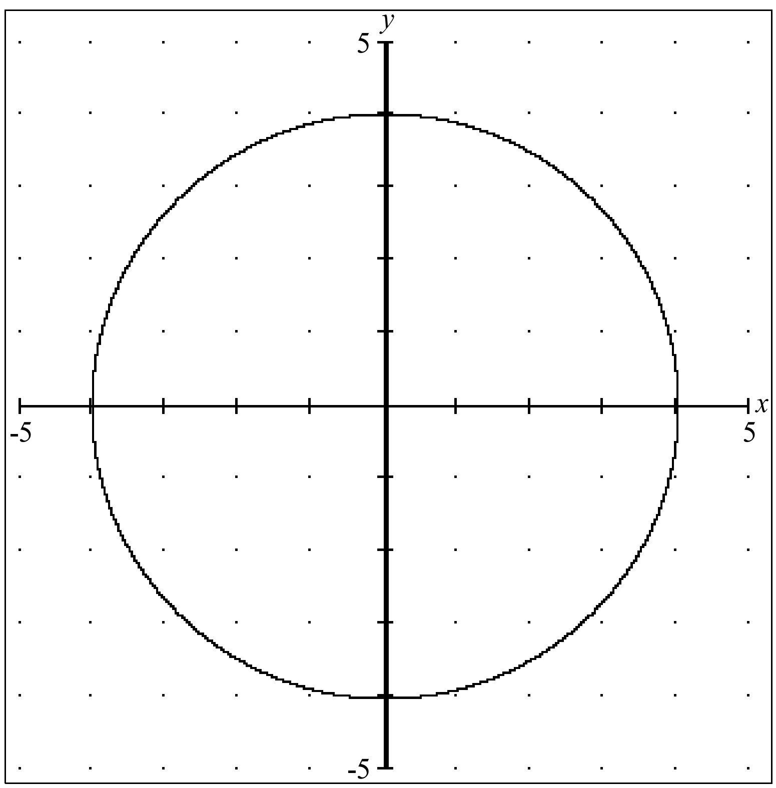


Fig.27

Basic Shapes

We shall use GrafEq to mechanically display the changes in a graph's appearance due to changing the defining equation. We first learn the 'basic' shapes associated with certain equations.

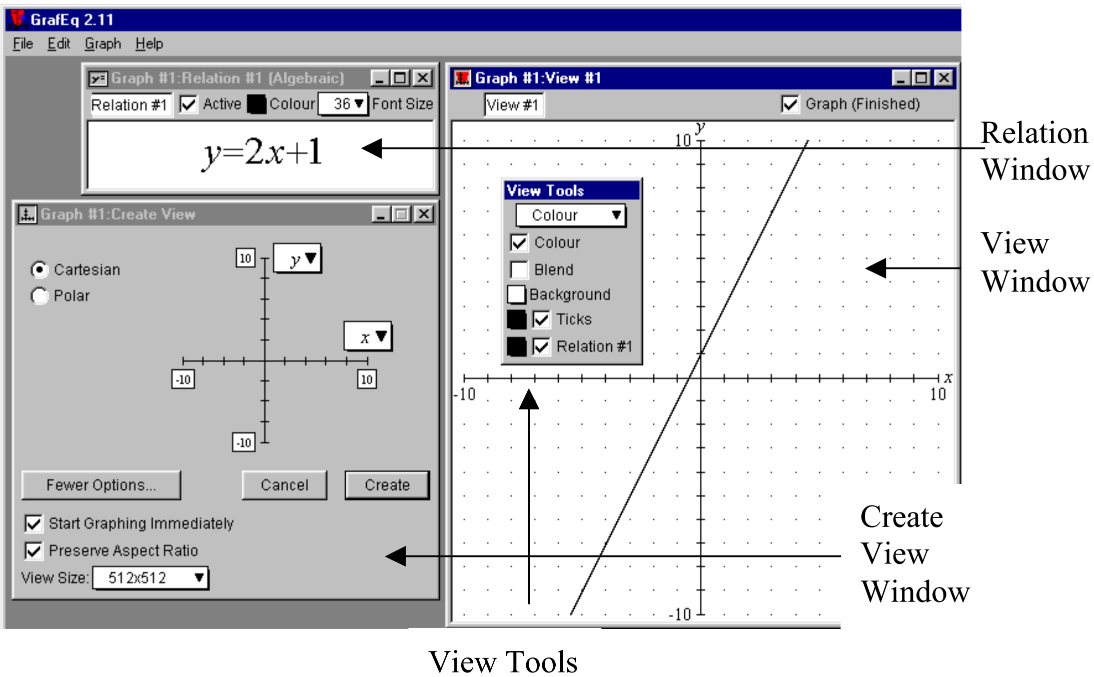


Fig.28 The Three GrafEq Windows

The Linear Function

In the Fig.28 above we see the three key GrafEq windows: the Relation window: used to enter the relation to be graphed - the Create View window: used to specify the bounds of the coordinate system (defaulted to ±10 for both x- and y- dimensions) - and the View window in which the graph is displayed.

1. Launch GrafEq by mouse-clicking on its icon.

2. Mouse-click (once) or use the space-bar (twice) to clarify and then clear the title screen.

3. In the initial relation window, enter the equation: y=2x+1

4. Accept the defaults (x: ±10, y: ±10) and view by clicking on *Create*.

5. Copy the computer graph to the coordinate system on the following page.

Note: if the dots do not appear on your screen: - select Ticks in the View Tools window and specify marks; then ensure that grid/dots are turned on. If the dots are not 1 apart, it may be necessary to select parameters and respecify the mark density.

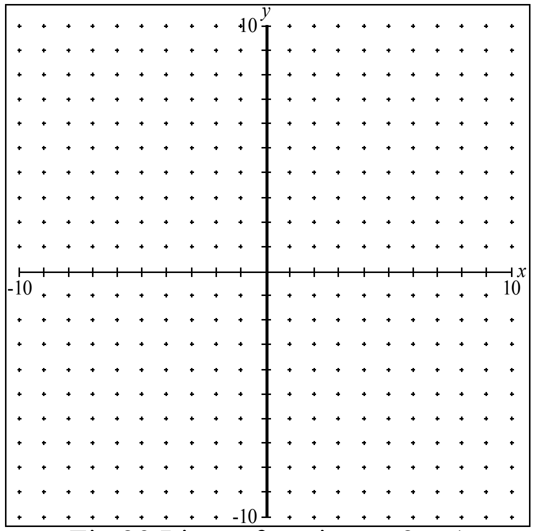


Fig.29 Linear function y=2x+1

The linear function y=2x+1 is a line with y-intercept: _____ and slope: ______.

Close the graph (under File).

The Quadratic Function (Parabola)

1. Select New Graph (under File)
2. Enter $y=x^2$ (Use either '^' or the up-arrow <↑>to exponentiate)
3. Accept the defaults in the Create View window.
4. Copy the graph carefully to the coordinate system below.

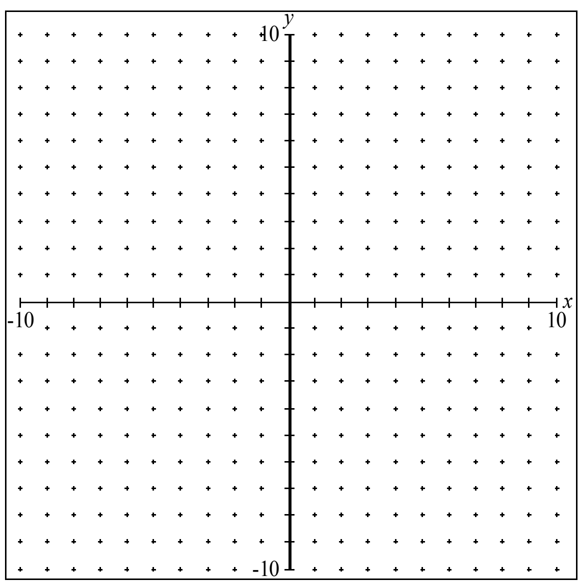


Fig.30 quadratic function $y=x^2$

The quadratic function $y=x^2$ is a parabola with vertex at (0,0). Other key dots are (-1,1) and (1,1). Emphasize these three dots by darkening them.

5. Close the graph (under File)

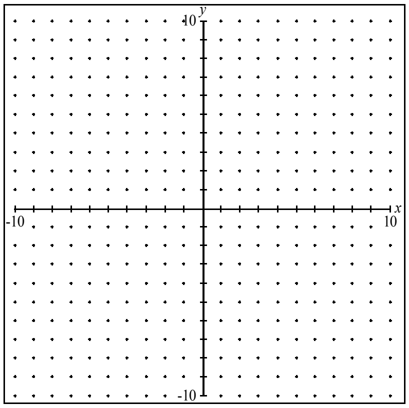
The Square Root Function

1. Select New Graph (under File)

2. Enter $y = \sqrt{x}$ (Use the $\sqrt{\ }$ button)

3. Accept the defaults in the Create View window.

4. Copy the graph carefully to the coordinate system in Fig.31 below.



The Square Root function $y = \sqrt{x}$ starts at (0,0) and passes through (1,1). Emphasize these two dots. The shape is a half-parabola.

Fig. 31 the square root function $y = \sqrt{x}$

5. Close the graph.

The Reciprocal Function

1. Select New Graph (under File)

2. Enter y=1/x

3. Accept the defaults in the Create View window.

4. Copy the graph carefully to the coordinate system below.

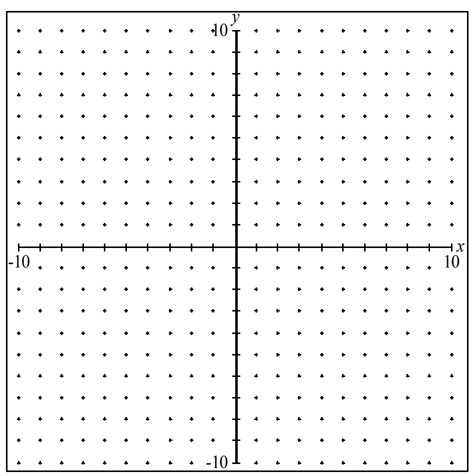


Fig.32 the reciprocal function y=1/x

The reciprocal function y=1/x consists of two curves, in quadrants 1 and 3. The key points to emphasize are (1,1) and (-1,-1).We call the x- and y-axes asymptotes because the curve approaches these lines.

5. Close the graph.

The Exponential Function

1. Select New Graph (under File)
2. Enter $y=2^x$
3. Accept the defaults in the Create View window.
4. Copy the graph carefully to the coordinate system below.

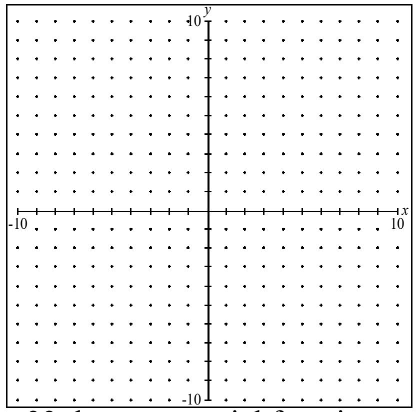


Fig.33 the exponential function $y=2^x$

The exponential function $y=2^x$ has a y-intercept at (0,1) and passes through (1,2). Emphasize these dots. Is there an asymptote? _________ What is it? _____________________

5. Close the graph.

The Cubic Function

1. Select New Graph (under File)
2. Enter $y=x^3$
3. Accept the defaults in the Create View window.
4. Copy the graph carefully to the coordinate system below.

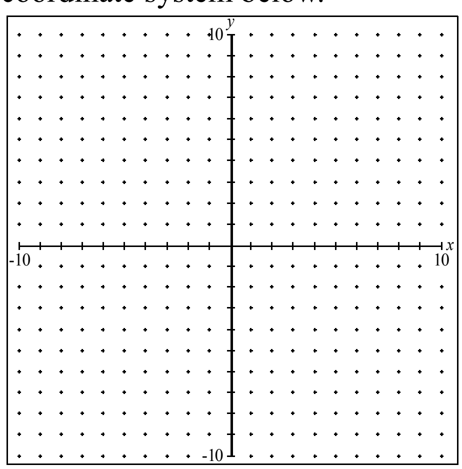


Fig.34 the cubic function $y=x^3$

The cubic function $y=x^3$ is 'S' shaped and passes through key points (-1,-1), (0,0) and (1,1). Emphasize these 3 dots.

5. Close the graph.

The Absolute Value Function

1. Select New Graph (under File)
2. Enter y=|x|
3. Accept the defaults in the Create View window.
4. Copy the graph carefully to the coordinate system below.

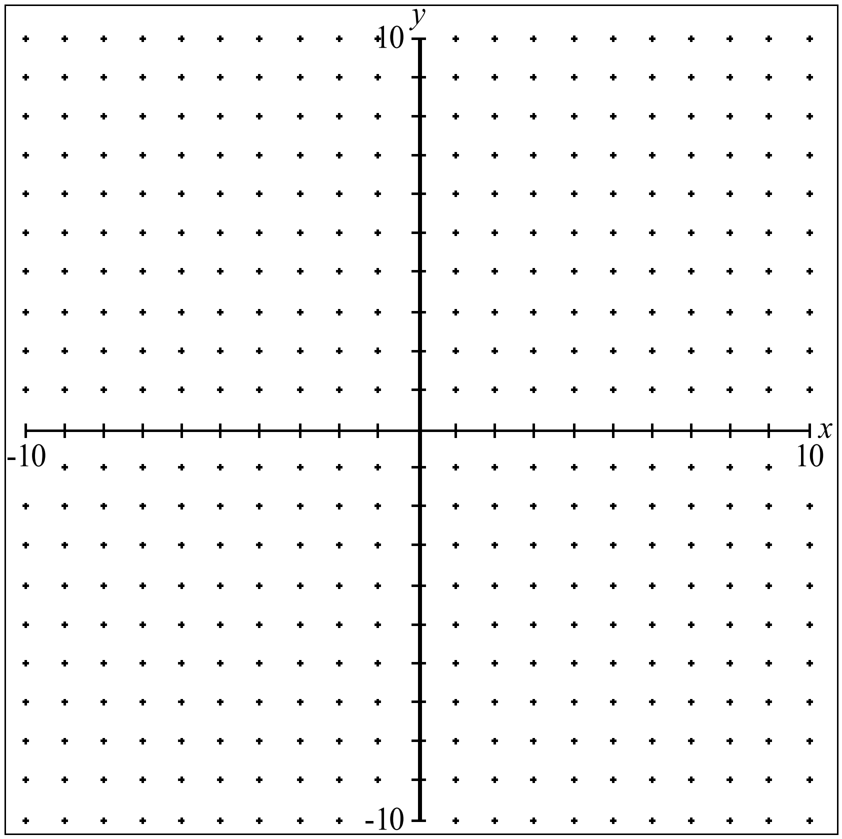


Fig.35. the absolute value function y=|x|

The absolute value function y=|x| is a 'V' shaped curve that passes through key points (-1,1), (0,0) and (1,1). The vertex is (0,0). Emphasize these three dots. What is the slope of the 2 sides? ______________

5. Close the graph.

Spend a few minutes examining these previous seven functions and their graphs. You should be able to draw each 'free-hand' by remembering the 'key dots'.

The TRANSLATION Transformation

What is the effect on the graph of a relation if we replace the x-coordinate with x-k , where k is a constant?

You will be able to easily answer this question with GrafEq.

1. Launch GrafEq by clicking on its icon.

2. Create a ‘session’ working environment as follows: Once the Relation window appears, before entering any equation, under File, select Preferences. From the pull-down list titled General, select Views. Set ‘display ticks’ to immediately. From that pull-down list (Views) select Ticks. Select Defaults under Controls. From the list titled General, select Dots and turn dots ON. From Dots, select Density and specify dense. Click on OK. From now on, until you quit GrafEq, the dot-ticks will appear immediately.

3. In the relation window, enter $y=x^2$. Your graph will appear as below in Fig.36. Note the 3 key dots (-1,1), (0,0), and (1,1). Now, under Graph, select New Relation.

Enter the relation $y=(x-2)^2$. You will see both graphs- the original and the 2nd relation. Carefully copy the 2nd graph to Fig.36 below. Note the new location of the 3 dots (see arrows).

Re-select New Relation, under Graph. Now, enter a 3rd relation: $y=(x+3)^2$. Copy its graph to Fig.36 below.

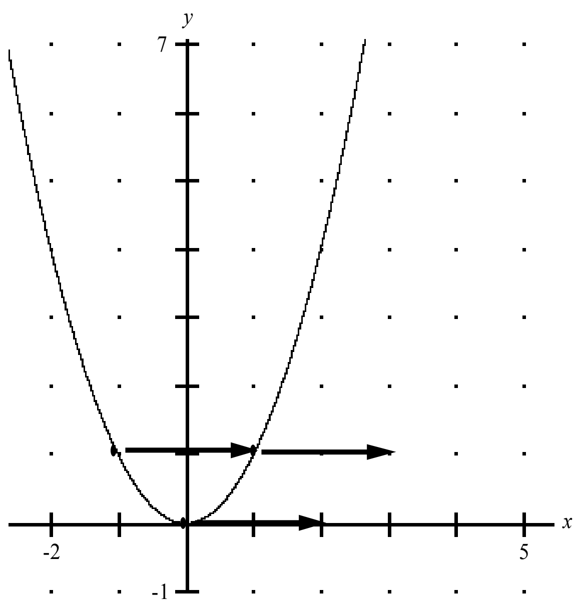


Fig. 36. quadratic relation

What is the effect of replacing x with x-2? ____

What is the effect of replacing x with x+3? ____

Summary: The effect of replacing x with x-k is a TRANSLATION (k to right if k is positive; k to the left if k is negative)

Close the graph. On Fig.37 below, draw the following graphs: $y = \sqrt{x}$, $y = \sqrt{x-4}$, $y = \sqrt{x+5}$

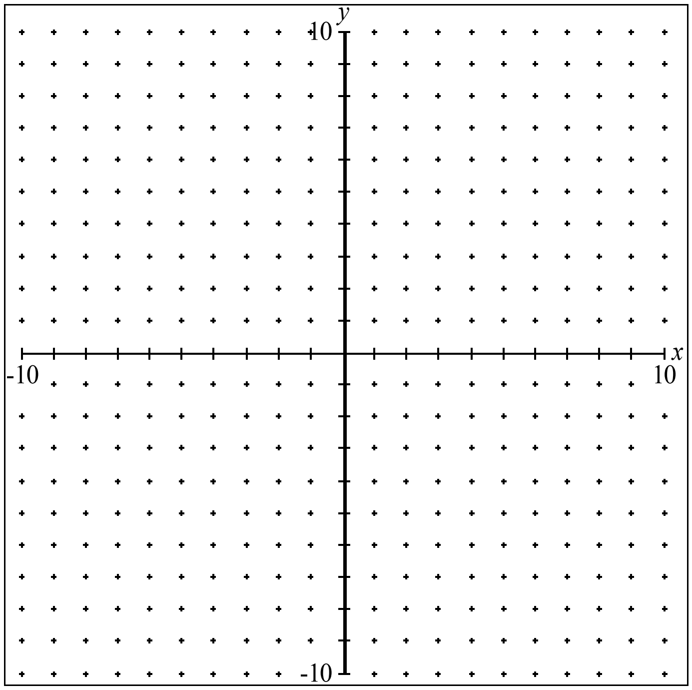


Fig.37. Square Root functions

Clearly mark the 2 key dots (0,0) and (1,1) on the basic curve $y = \sqrt{x}$ Use arrows to show the new locations of these key dots.

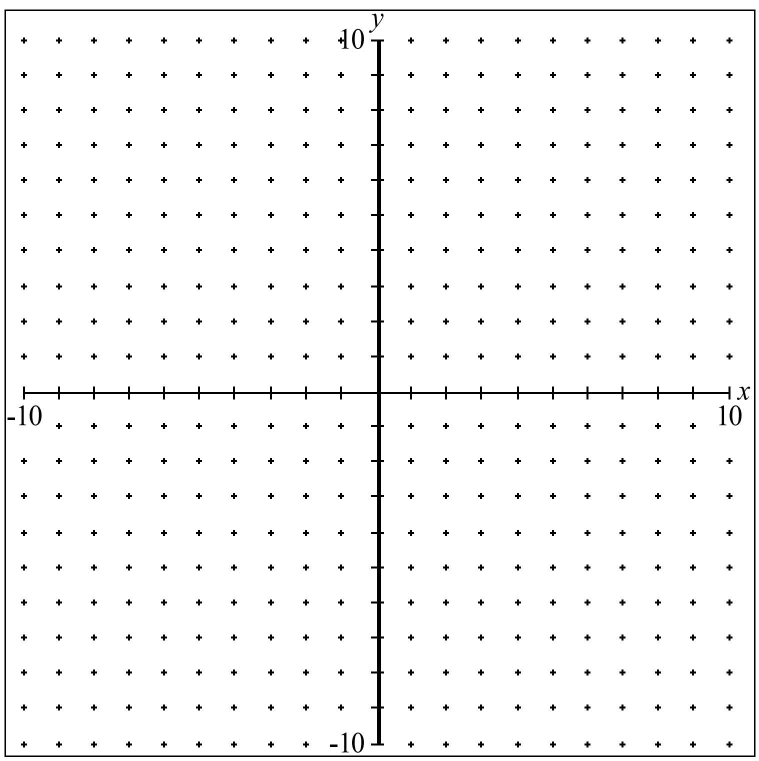


Fig.38 $y=x^3$ $y=(x-5)^3$ $y=(x+6)^3$

Use arrows to show the new locations of the key dots (-1,-1) (0,0) and (1,1) above.

On Fig.39 below, draw the following graphs: $y=2^x$ $y=2^{x-2}$ $y=2^{x+3}$

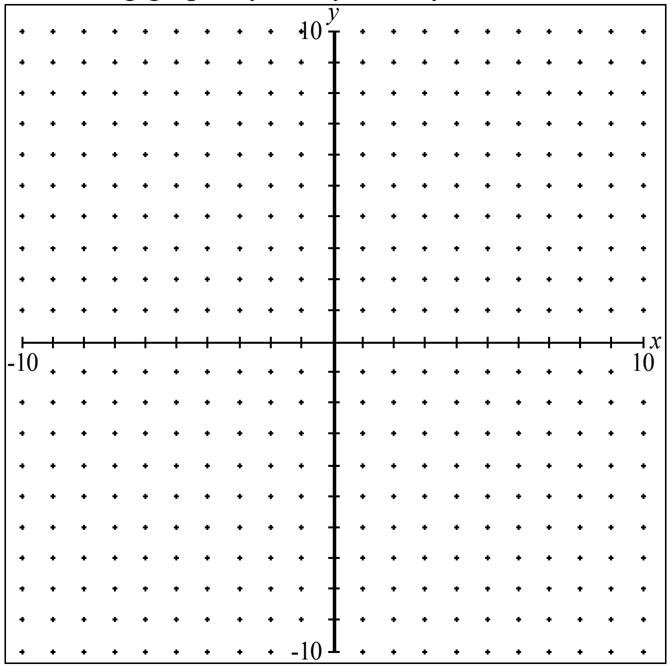


Fig.39. Exponentials: $y=2^x$ $y=2^{x-2}$ $y=2^{x+3}$

Clearly mark the 2 key dots (0,1) and (1,2) on the basic curve $y=2^x$. Use arrows to show the new locations of these key dots.

Complete all graphs for Fig.40 below.

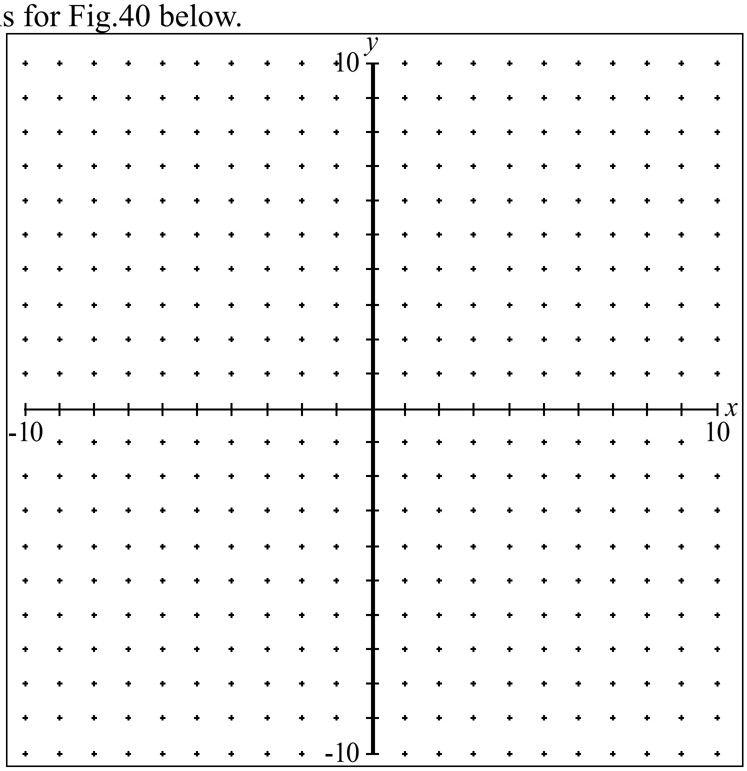


Fig.40 $y=|x|$ $y=|x-5|$ $y=|x+7|$

Use arrows to show the new locations of the key dots (-1,1) (0,0) and (1,1) above.

On Fig.41 below, draw the following graphs: $y=2^x$, $y-1=2^x$, $y+3=2^x$

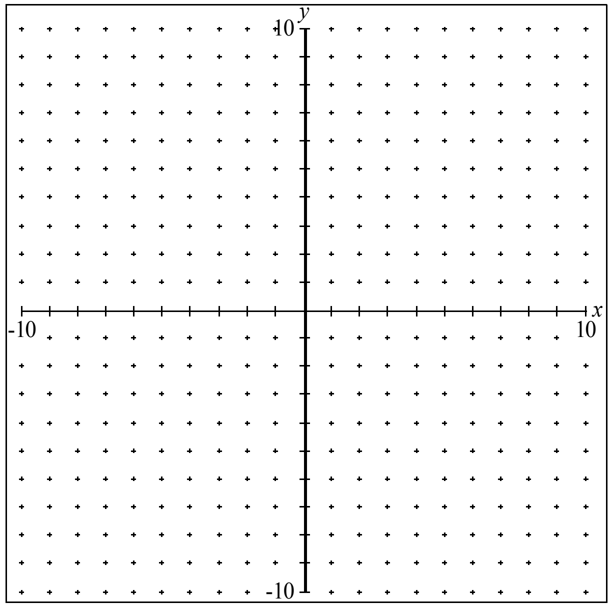


Fig.41 Exponentials $y=2^x$, $y-1=2^x$, $y+3=2^x$

Clearly mark the 2 key dots (0,1) and (1,2) on the basic curve $y=2^x$. Use arrows to show the new locations of these key dots. What is the effect of replacing y with y-1?

What is the effect of replacing y with y+3?

Complete all graphs for Fig.42 below.

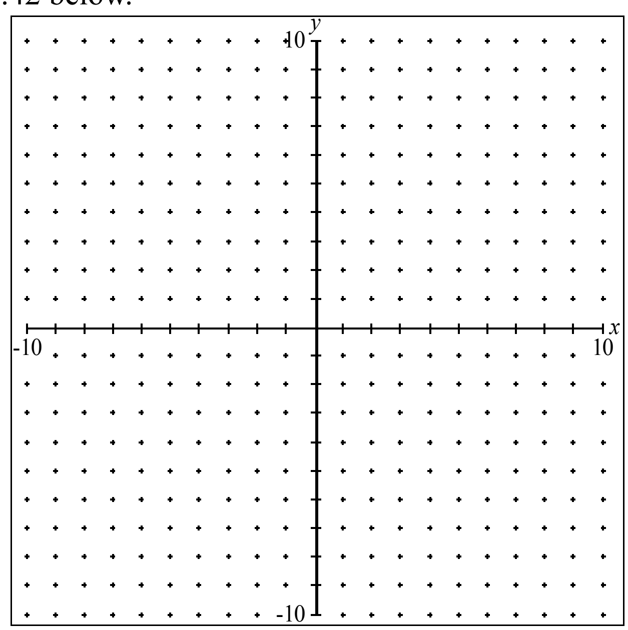


Fig.42 $y=1/x$ $y-3=1/x$

Use arrows to show the new locations of the key dots (-1,-1) and (1,1) above.

Now, let us combine both horizontal (x) translations and vertical (y) translations.

Complete the graphs for Fig.43 below

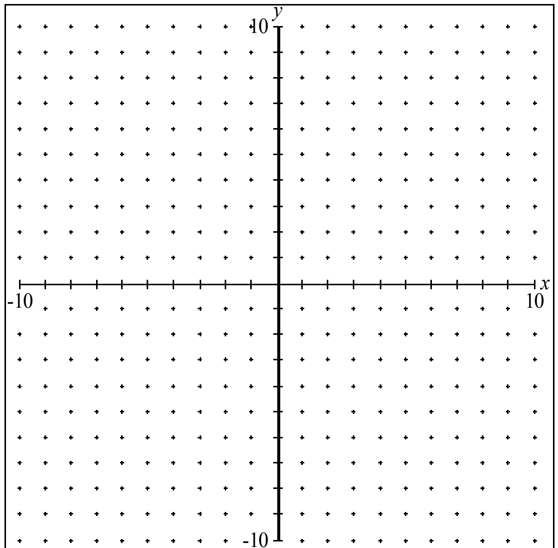


Fig.43 $y=x^2$ $y-2=(x-3)^2$

Use arrows to show the new locations of the key dots (-1,1) (0,0) and (1,1) above. If you did not use GrafEq to complete Fig.43, then check your solution with GrafEq.

Complete the graphs for Fig.44 below

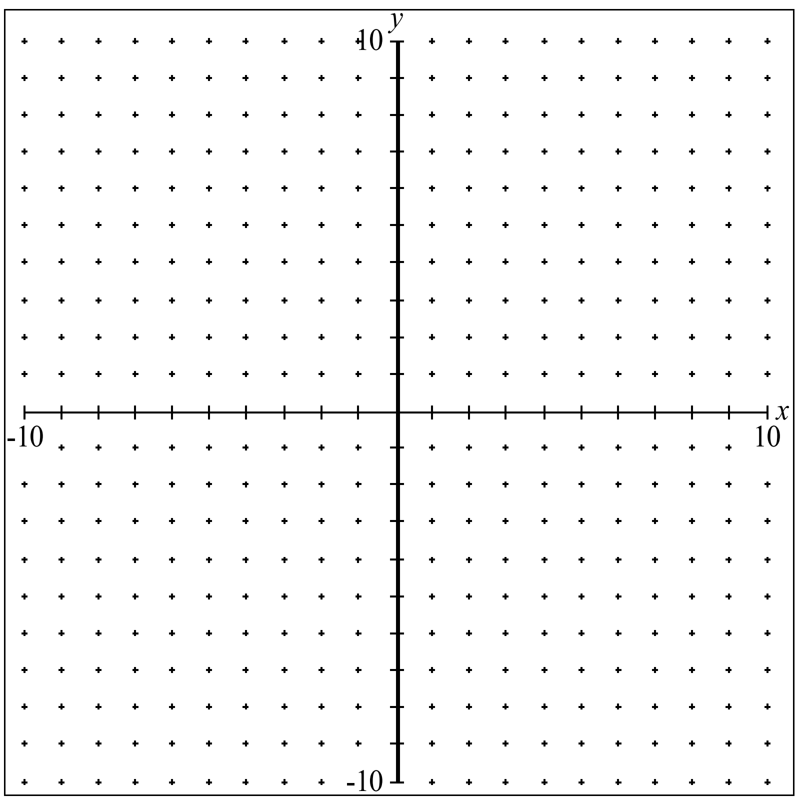


Fig.44 $y=|x|$ $y+2=|x-5|$

Use arrows to show the new locations of the key dots (-1,1) (0,0) and (1,1) above. If you did not use GrafEq to complete Fig.44, then check your solution with GrafEq.

The DILATION Transformation

You have learned the effect of replacing x with x-k, or replacing y with y-k. You noted there was no change in shape (a dilation) but only a change in location (a translation). What replacement will cause a change in shape?

Complete the graphs in Fig.45 below.

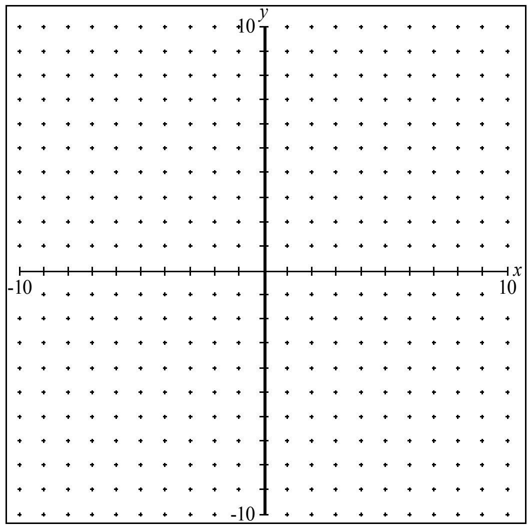


Fig.45 $y=x^3$ $y=(x/4)^3$

Note the original 3 key dots (-1,-1) (0,0) and (1,1). The horizontal distance from the y-axis has quadrupled. I.e. (0,0) is now at (4∗0,0), (-1,-1) is at (-4,-1) and (1,1) is at (4,1). What would the graph of $y=(x/.5)^3$ look like? Without using GrafEq, lightly sketch your answer. Check your answer with GrafEq. Complete the graphs in Fig.46 below.

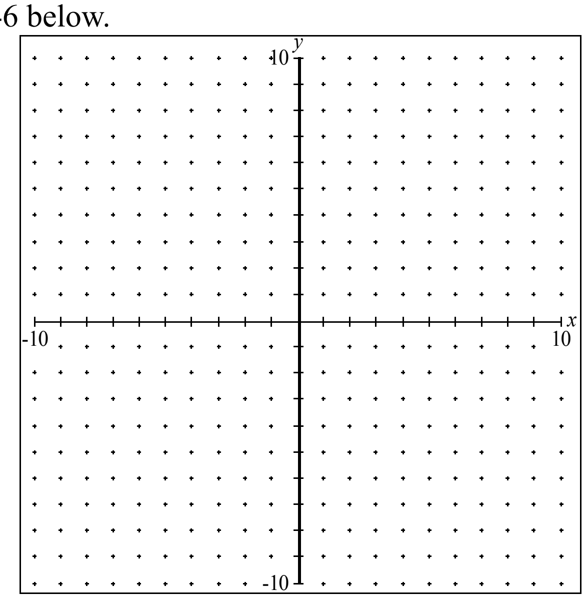


Fig.46 $y = \sqrt{x}$ $\frac{y}{5} = \sqrt{x}$

Note the original 2 key dots (0,0) and (1,1). The vertical distance from the x-axis has quintupled.

Ie (0,0) is now at (0,5∗0) - and (1,1) is at (1,5).

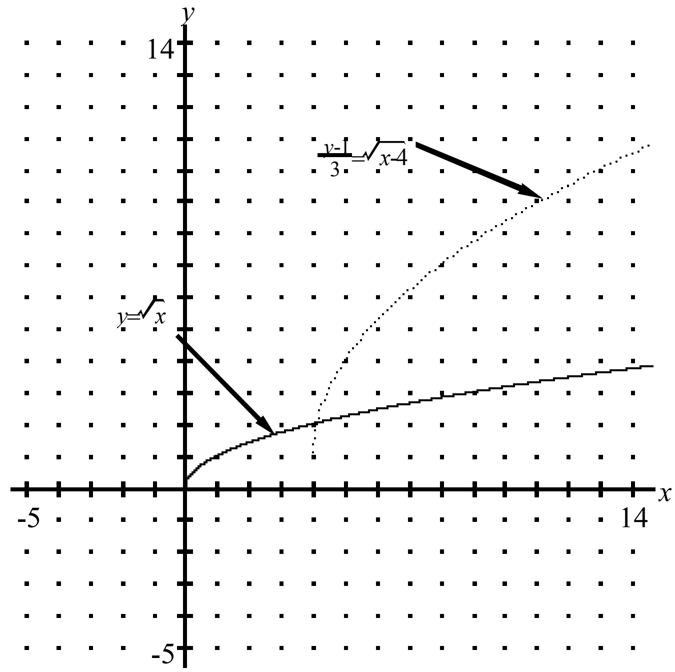


Fig.47 $y = \sqrt{x}$ and $\frac{y-1}{3} = \sqrt{x-4}$

In Fig.47 above, mark the key dots (0,0) and (1,1) on the original basic curve:

Explain how the new graph is derived from the original.

____
____
____
____
____
____

Summary:

We have studied two sorts of transformations: translations and dilations - a translation or 'shift' is effected if we replace x with x-k or y with y-k. For instance, replacing x with x-4 will shift the graph 4 to the right. Replacing x with x+2 will shift the graph 2 to the left. Similarly, replacing y with y-3 : 3 up; replacing y with y+6: 6 down. Note that altering x causes a change in the horizontal (x-direction) and altering y causes a vertical change.

A dilation or 'stretch' is effected if we replace x with x/k. For instance, replacing x with x/3 will triple each point's horizontal distance from the y-axis. And replacing x with x/.5 (or 2x) will halve each point's x-coordinate. The effect of equivalent y-changes will be similar.

EXAMPLE: Graphing $y = \sqrt{2x+1}$ We start with the graph of $y = \sqrt{x}$ with key points at (0,0) and (1,1). then we replace x with x+1 (a shift of 1 to the left): $y = \sqrt{x+1}$ and then we replace x with 2x (a halving of the x-coordinates).

So (0,0) → (-1,0) → (-.5,0) That is, (0,0) ends up at (-.5,0).

And (1,1) → (0,1) → (0,1) That is, (1,1) ends up at (0,1). Study the graph below carefully.

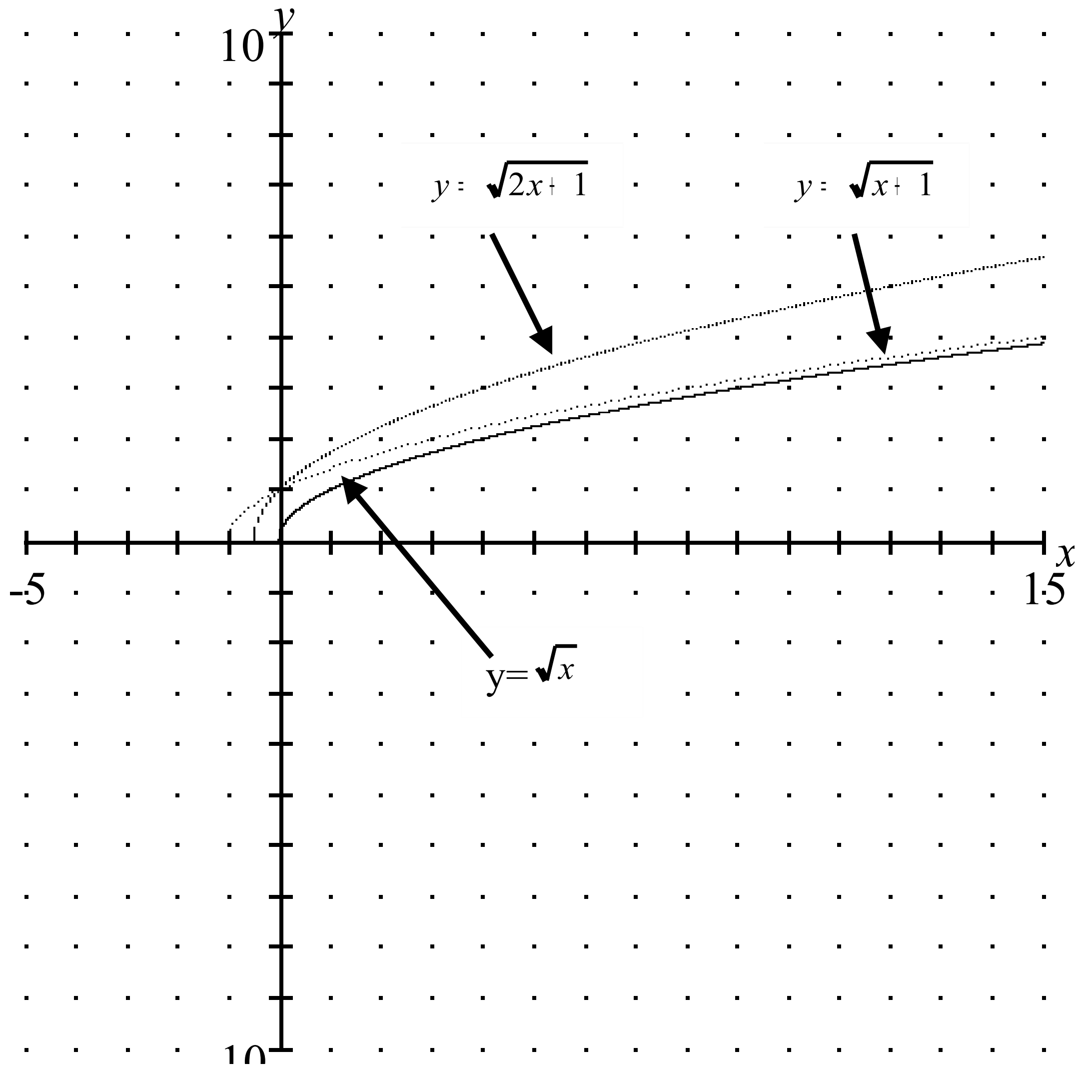


Fig.48

You will see how the original curve ($y = \sqrt{x}$) has shifted 1 to the left, then dilated by one-half horizontally.

Calculus

The Slope of the Tangent Line

Teacher Demo

(To be used with a projection device or lan screen sharer or large screen TV)

Concepts to which students have had prior exposure:

- definition of slope of a line
- tangency
- limits (intuitive)

PROBLEM : Determine the slope of the line tangent to $y=x^2$ at the point (1,1).

METHOD : Determine the slopes of segments whose end points are on the graph (one of which is at (1,1)) and study what happens to the slopes as the other point approaches that point.

1. Launch GrafEq by double - clicking on its icon.

Click (twice) to dispose of the title screen.

2. Enter the equation : $y = x^2$ in the relation window. Use the <↑> key to commence the exponent and enter <Enter ↩> once the equation is properly entered.

3. In the create view window set x to range from -2 to 2; set y to range from -1 to 3.

Click on *Create*, to invoke the view window.

4. Under Graph (in the menu bar), select New Custom Ticks. Enter the following custom ticks#1:

x=1. Activate by entering <Enter ↩>

Note the vertical tick-line (x=1) in the view window.

5. Under Graph, re-select New Custom Ticks and enter: y=1 <Enter ↩> There are now two tick lines, intersecting at (1,1) on the parabola.

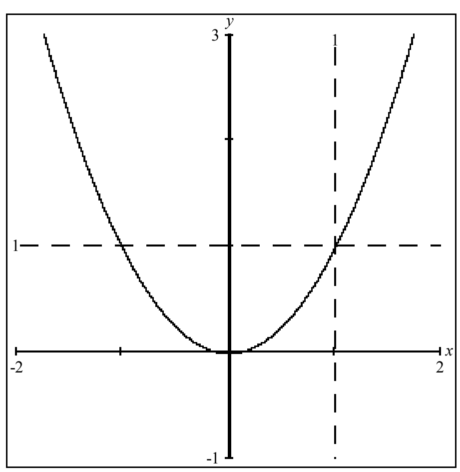


Fig.49

6. Ensure that the view windows is active (front most, with darkened title bar) and enter <z> - this is a one-key shortcut to invoke the zoom mode.(It could have also been done by selecting zoom in the pull-down mode list with default title “General”.)

7. Carefully place the zoom frame over the intersecting ticks at (1,1) and click. Repeat this process once or twice more (until the parabolic curve appears ‘straight’)

8. Enter <2> to invoke the two point mode. Use the mouse to drag the point ‘A’ onto the intersection of the ticks at (1,1). (Press the mouse button over ‘A’ and drag ‘A’ onto (1,1).)

Similarly drag point ‘B’ to another point on the curve. (see Fig.50 below.)

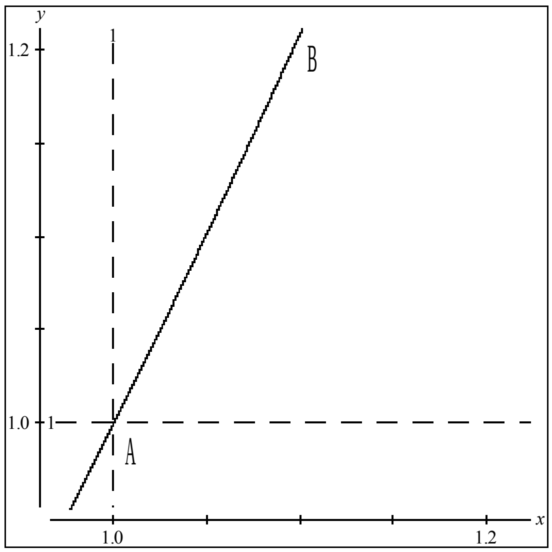


Fig.50

In the two point sub-mode title box (defaulted to ‘Distance’), select Slope. The slope of segment AB is approximately 2. If you were to repeat this process (steps 6 and 7 above) a few more times, and then step 8, to determine the slope, you would approach the value 2.00 as the curvature approaches linear.

A similar process may be used to approximate the slope of the tangent of any graphable curve.

For the teacher

How to Incorporate GrafEq Graphs into Text Documents

If you wish to incorporate graphs produced by GrafEq into text documents such as worksheets or tests, there are two distinct methods. The simplest method is to simply print the graph via GrafEq's print (New page under Graph) option and use a photocopier with enlarge/reduction capability to resize the graph and then physically paste the graph onto the text document for further copying. This technique permits using the high resolution capability of GrafEq to print the graph and the resolution will not be 'lost' by photoreduction. (the dots will simply be smaller)

The alternative technique is to transfer the graph internally to a word processor with graphic capabilities.

The method is simple:

1. Create the graph with GrafEq. When the desired graph is in the current viewport:

2. Select *Copy* (specified to image) under Edit. This copies the active viewport to the clipboard. You may need to experiment with the various choices – 'drawing' or 'image' to determine which works best with your word processor.

3. Quit GrafEq and run a word processing program -or if your computer has adequate memory, you will be able to run both programs concurrently.

4. Import the GrafEq screen via the paste (special) command - according to the requirements of the WP program.

You can similarly copy and paste from a GrafEq *Relation* window. The 'text' option is handy for sending a relation in an e-mail message.

Creating Graph Paper with GrafEq

In addition to its graph-plotting capabilities, GrafEq enables the user to fine-tune the ‘background’ coordinate system - with its ticks. There are at least two reasons for the teacher to create his/her own coordinate systems: firstly to make printouts for student use and secondly to create templates for use in subsequent graphing sessions.

Creating customized graph paper

The user can create custom paper as follows:

1. Launch GrafEq.

2.Leave the relation window empty, and under Graph select New View..

3. In the ‘Create View’ window, select an appropriate view size - say 768X768. (The larger the view size - the smaller the ‘dot’ size on the print out.) Turn off Display view in color. Then click on ‘Create’ or press <Enter ↩>.

4. In the resultant View window enter <t> to access the ticks mode, and activate the ticks by clicking the show ticks box if it is not already ‘checked’. Then, from the pull-down list select ‘marks’ and activate the dots by selecting ‘dots’ in the grids sub-list. Then select the axes sub-list and check ‘arrowheads’. Then change ‘marks’ to ‘parameters’ and specify dense for mark density If the Graph paper seems OK, the user will commence the creation of a page for print-out.

5. Under the ‘Graph’ menu, select ‘New Page’. If additional text is desired, select the text (A) tool and enter the text. The text may be easily edited or re-positioned. Once the display is acceptable, select the printer tool and proceed on to print the page. Note: if it is desired to print, say, six smaller copies of the coordinate system on a single page: select the original view with the pointer tool and resize it to 25% Then select the View tool and specify View#1 at 25% and place another copy of the view in an appropriate location. More copies of View#1 can be then selected and positioned in order to complete the page.

Creating a template for use in trig demos

If you anticipate creating trig graphs in class demo sessions, it may be desirable to create an appropriate coordinate system in advance. It can be done easily as follows:

1. Launch GrafEq.

2. Enter a suitable basic equation such as y=sinx.

3. In the Create View window specify a domain such as x: -4 to 7 and y:-4 to 4 and Create the view.

4. Enter <t> to access the ticks mode, and specify appropriate marks and parameters from the pull-down list. Select New Custom Ticks under the Graph menu. Select pre-defined ticks and specify multiples of / $\pi/2$ excluding x= 0. Once in the tick window, place the cursor to the right of k/2 - delete the 2 - and replace it with a 6. Click the cursor to the right of the equation $y=k/6\ \pi$. Use the leftmost ticks pull-down list and select the alternating pattern. Upon pressing <Enter ↩> you will be presented with a view containing vertical dotted lines at multiples of $\pi/6$.

5. Reselect New Custom Ticks under Graph, and respond *No* to pre-defined ticks. Enter the following two-constraint tick definition: (Use <Tab> to go from the first constraint to the second. Use the easy

button to access the ± operator.)

y=k/2

k={±1, ±2, ±4, ±6}

6. Use the second (from left) pull-down tick list and select the middle option (dashed). Upon pressing <Enter ↩> your view will be augmented by horizontal dashed lines at y= ±.5, ±1, ±2,±3.

7. If the template is satisfactory, Save it (under File) as a skeleton or full graph. Once saved, with an appropriate file name - like ‘trig paper’, you can, in subsequent GrafEq sessions, ‘Open’ the template and use it to graph y=2sinx, y=sin(x-π/2) etc. and note the differences with the basic sine curve. In the View Color mode, you can turn color off, deactivate the sin relation, and go on to print a Page containing the ‘blank’ B&W trig graph paper.

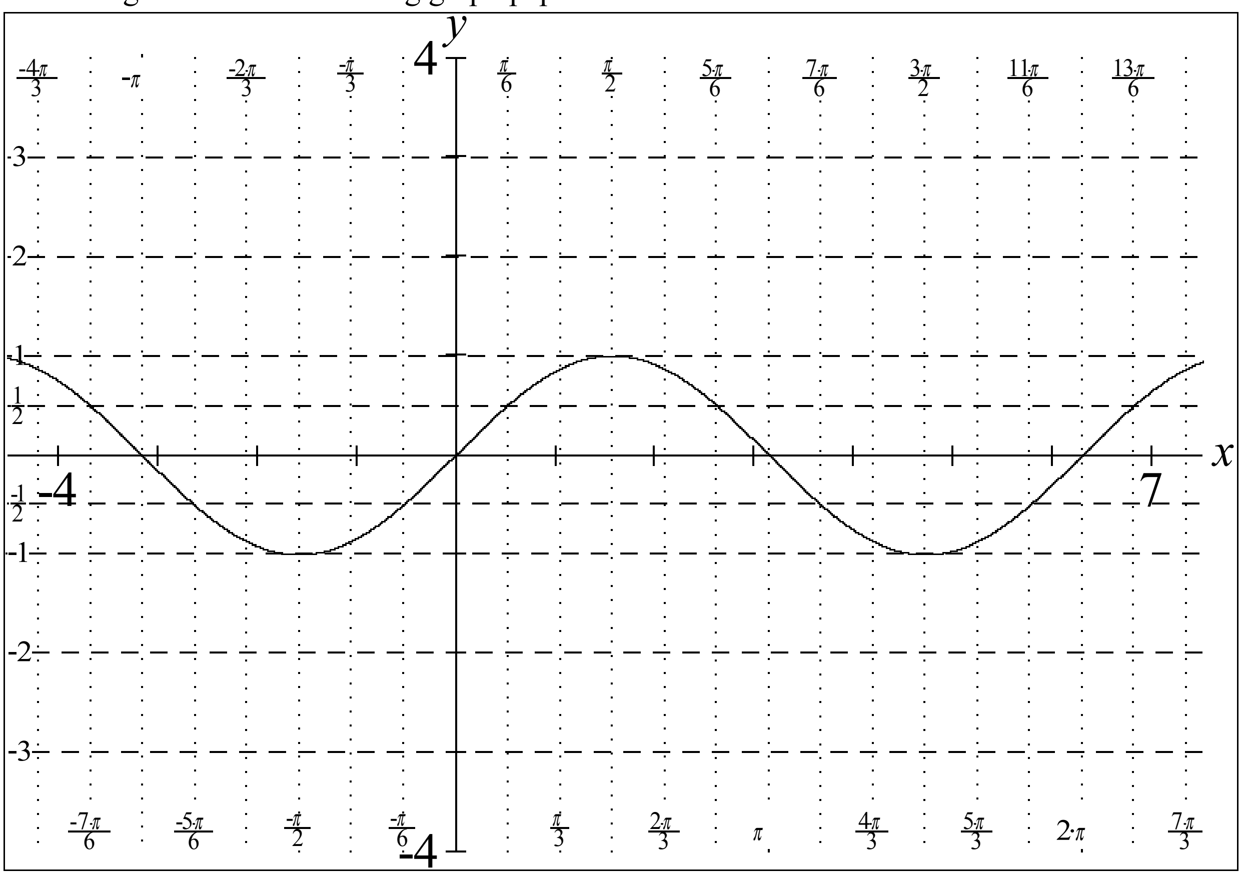


Fig.51. sample template for trig demos

The template in Fig.51 can be particularly useful in answering questions like “What if we now graph y=sinx+2 , or y=sin(x-2)?” The resultant graphs can overlay the original ‘basic’ curve

The Plot Thickens

Students of senior secondary mathematics now have easy access to electronic plotting devices: either graphing calculators or computer applications. Both students and their teachers may benefit from an understanding of how these devices work - from a mathematical perspective. This paper will examine some of the problems inherent in such devices - and demonstrate possible solutions. In addition, the writer will present some secondary level mathematics that can be easily pursued via these devices. All illustrations are produced by GrafEq™ on a Macintosh computer. The reader is encouraged to read this article with a calculator or graphing program at hand in order to test out the claims in the article. All future references to 'grapher' will mean graphing calculator or computer utility.

If we examine a grapher display, we will note that it is simply a rectangular array of 'dots' or pixels, in computer jargon. On a monochrome device these dots will be either 'on' or 'off'. We must bear in mind that these pixels, although small, have area - and are therefore better perceived as small rectangles rather than points. The assumption of the user is that 'on' pixels contain solution points and 'off' pixels do not. If we plot $x^2+y^2 = 64$ we will see a 'circle' whose appearance reflects the inherent difficulty in plotting a curve on a rectangular grid. 'Jaggies' appear more pronounced at various sections of the curve due to the rectangularity. We can attempt a further illustration of the difficulty of pixel-plotting by plotting a relation whose graph is a single point: a circle of radius 0: $x^2+y^2 = 0$. The plot *ought* to be a single pixel at best - four pixels at worst. If the axes run between rows and columns of pixels, 4 pixels ought to be 'on'; if the axes run through the pixels, a single pixel ought to be 'on'.

Some graphers will have difficulty in plotting a single-point relation. The reason for this is the technique used to determine solution points : sampling. Many programs use this method, which is analogous to how we might plot a graph manually by point-plotting: the program subdivides the visible domain into many intervals, then at each x-value calculates the corresponding y-value. Consecutive points are connected. This method is very fast and often adequate. However, in a single-point situation, that specific x-value may possibly never be encountered and the pixel containing it will be left 'off'.

There are some other difficulties with the sampling technique: non-functional relations wherein more than 1 y-value corresponds to the x-value - curves with very narrow 'spikes' - curves with very dense periods - and non-curvilinear relations . The reader is invited to try graphing the following:

$x=\sin y$	not a function – ought to be a vertical sinusoidal.
$xy=.00001$	a hyperbola with a narrow 'spike' – overlaps the x- and y- axes.
$y=\sin 500x$	a very dense curve – apparently solid – yet actually sinusoidal.
$y=\lvert y\rvert$	solid – quadrants I and II.
$x^2+y^2 = x^3+y^3$	a curve *plus a single point.*

Another troubling situation for the mechanical grapher: discontinuities. Most sampling programs will check for excessive distances between 'adjacent' points before connecting them - this will preserve the discontinuities in hyperbolas and tangent and secant functions. The case of point discontinuities is more troublesome. The teacher at the chalkboard will often interrupt a curve with a hollow dot. The grapher may nevertheless determine solutions within the pixel containing the discontinuity. The reader is invited to try graphing $y = (x^2-1)/(x-1)$. The proper graph is that of the line: $y=x+1$ minus the point (1,2). And this poses a real problem: the pixel containing (1,2) contains many solutions - therefore it should be 'on'. Yet if it is on, the viewer will be unaware of the existence of the discontinuity. The reader might suggest : "if there exists even a single discontinuity within a pixel, then that pixel should remain off" But this suggestion has its own pitfalls - for instance, a curve with 1 discontinuity in every pixel of its display. Such a curve would not be visible at all if the grapher followed this process. There may be a way, however, and that is due to the dynamic aspect of the electronic grapher. One possibility might be a 'blinking' pixel - signifying the existence of a discontinuity. The preceding discussion only has merit, however, if the grapher is 'aware' of the discontinuity - and that presupposes a symbolic capability, such as the ability to spot division by zero.

A relation to try: $y=x^2(x-1)/(x-1)$ (a parabola with discontinuity at (1,1))

Inequalities: these can be plotted by sampling: one method is to first establish the edge, and then 'shade' the appropriate side. But this may not work always: consider $y<\sec x$.

Relations to try:

$y<x$ (a half-plane)

$x^2+y^2>16$ (exterior of a circle)

The reader may by this time have noted another problem area: relation specification. If the grapher uses sampling it is most convenient to deal with an explicit relation: one in which a single variable is on the left side and an expression involving the other variable is on the other side (such as $y = 4x^3-2x + \log x$). Some graphers restrict the user to this format. Some will accept input in a more relaxed format - such as $x+y= y\sin x$ - because the grapher'sparser can rearrange the relation to an explicit form (in the preceding case: $y = x/(\sin x-1)$). Most convenient to the user, however, is a grapher's ability to process implicit relations - those which cannot be rearranged to an explicit form, such as $y\log yx^x=\cos x^y + y^y$. On a similar note, the reader should be aware of the display or format displayed. A grapher that displays standard math notation with proper subscripts, superscripts, division bars etc. provides valuable feedback to the user. A linear display, such as ((x-3)^2)/4)+((y+1)^2)/9=1 is indeed the equation of an ellipse - but not obviously so. Whereas the display below

$$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{9}=1$$

provides visible confirmation that the relation has been entered correctly.

Is sampling the only technique available? The answer is no - an alternative technique involves interval arithmetic, which is illustrated by the following one-dimensional example. Suppose we wish to solve $x^2=9$ over the domain 0-10. If $x=0$ then $x^2=0$ and if $x=10$ then $x^2=100$. Since $0<9<100$ we can deduce that a solution may exist in that interval. We can then subdivide the interval into sub-intervals: say 0-5 and 5-10. And now we can examine each of these intervals. If x is between 0 and 5 then x^2 is between 0 and 25 and 9 is within this range. If x is between 5 and 10 then x^2 is between 25 and 100 and 9 is not within this range. We may now discard the interval 5-10 since it contains no solutions, and go on to subdivide the interval 0-5. We can continue this process until we reach the desired precision of solution. The reader will note that this process is essentially subtractive - those parts of the domain not containing solutions are discarded. This contrasts sharply with sampling, which is essentially additive. Although the preceding example is one-dimensional, the method generalizes to multi-dimensional situations.

Logically, how does this subtractive interval arithmetic method compare to the additive sampling technique? The major advantage is that 'off' pixels correctly contain no solutions, although 'on' pixels may also incorrectly contain no solution. That is, there are no errors of omission although there may be errors of commission. For an example of interval arithmetic plotting see Fig.53. Sampling, however, permits errors of both types as is illustrated in Fig.52.

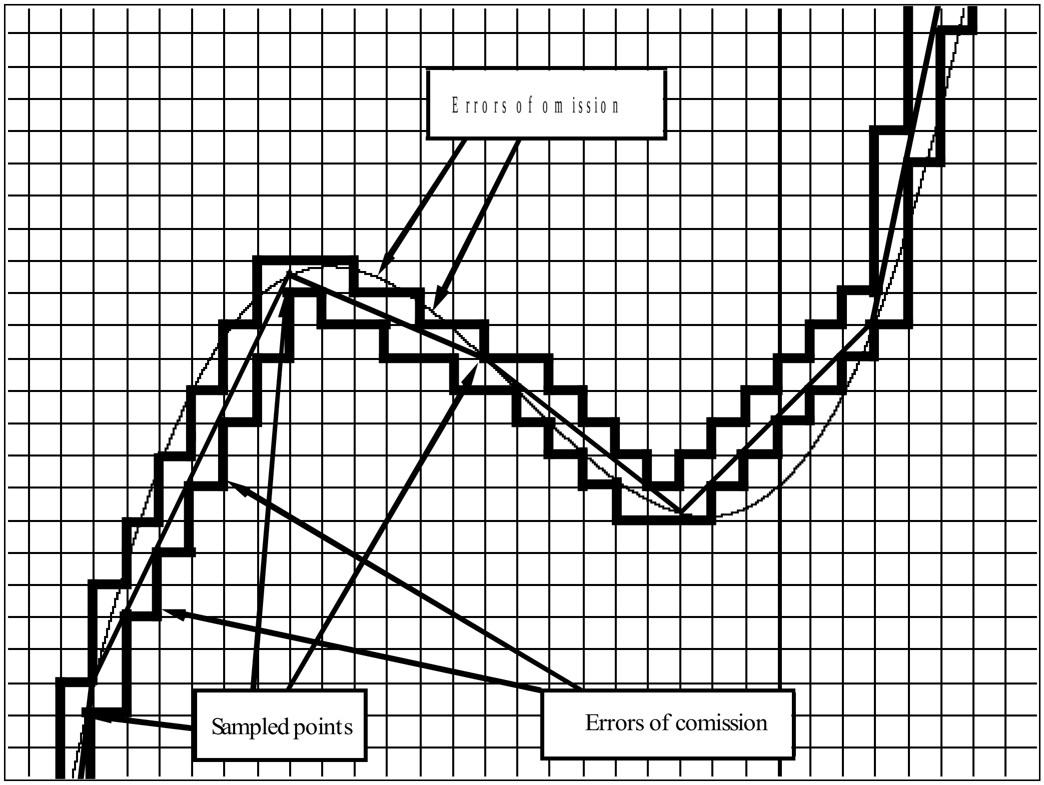


Fig.52 Potential errors due to sampling

A study of mechanical graphers leads to a number of interesting questions. One of which is: "Is there a mathematical relation for every visible graph?" If we interpret 'visible graph' to be a finite collection of 'on' pixels on a computer screen, the answer is yes. It can be accomplished as follows. Suppose we wish a relation to produce the two pixels containing, say, (1,3) and (-2,6). The relation: $(x-1)^2+(y-3)^2=0$ will produce the first pixel and the relation $(x+2)^2+(y-6)^2=0$ will produce the second. We can combine both of these relations as $[(x+2)^2+(y-6)^2][(x-1)^2+(y-3)^2]=0$ - the relation whose graph is both dots. We can thus build up any finite number of pixels by this method, the only practical consideration being whether the resultant relation exceeds the capability of the grapher's input. An implication of the preceding argument is that non-equivalent relations can produce identical pixel displays for any given domain and range.

The method of the preceding paragraph is based on the property: 'If ab=0 then a=0 or b=0' - well known to students of factoring. This property can also be used to graph a number of curves via a single relation. Suppose, for instance, we wish to plot a hyperbola and ellipse via a single equation. The following will fit the bill: $(xy-1)(x^2+4y^2-4)=0$. The first factor determines the hyperbola; the second determines an ellipse.

Analytic geometry in the secondary curriculum limits itself to a number of entities: lines, circles, ellipses, parabolas etc. But many interesting constructs are not discussed at all. The following can be plotted if we can determine their algebraic definitions:

Segment: $\sqrt{(x+2)^2+(y-4)^2}+\sqrt{(x-1)^2+(y-2)^2}=2\sqrt{2}$

Ray: $\sqrt{(x-3)^2+(y-4)^2}+2\sqrt{2}=\sqrt{(x-1)^2+(y-2)^2}$

Square: $|x|+|y|=5$

Star: $$\frac{r}{\sin .1\pi} = \frac{10}{\sin(.9\pi - \mathrm{mod}(\theta, .8\pi))}$$
$$0 < \theta < 4\pi$$

Quincunx:

Pentagon:

Confirmation of the first four above, and derivation of the last two is left as an exercise for the reader.

In closing, a number of general observations have been expressed about the mechanics and limitations of mechanical graphers. It seems apparent to the author that the advantages of such devices are two-fold - firstly they remove the drudgery of calculation required to produce a plot. But secondly, and more importantly, they afford a tool to stretch the student's mathematical imagination.

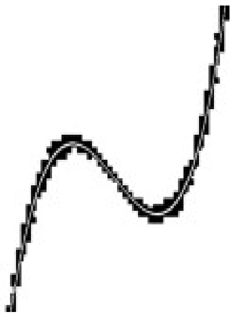


Fig.53 Curve being plotted by interval arithmetic process

Inequalities

Inequalities Without Inequality Signs

Enrichment lesson

Although GrafEq does support inequalities directly via the ‘<’, ‘>’ etc. signs, you may use the program to graph inequalities using only the ‘=’ sign.

Suppose you wish to plot the graph of $y<x^2-5$

Method: You can actually graph the whole region by using the following property:

$x>y$ if and only if $x-y = |x-y|$. This enables the use of the equation: $y-x^2+5=|y-x^2+5|$ to produce the following output.

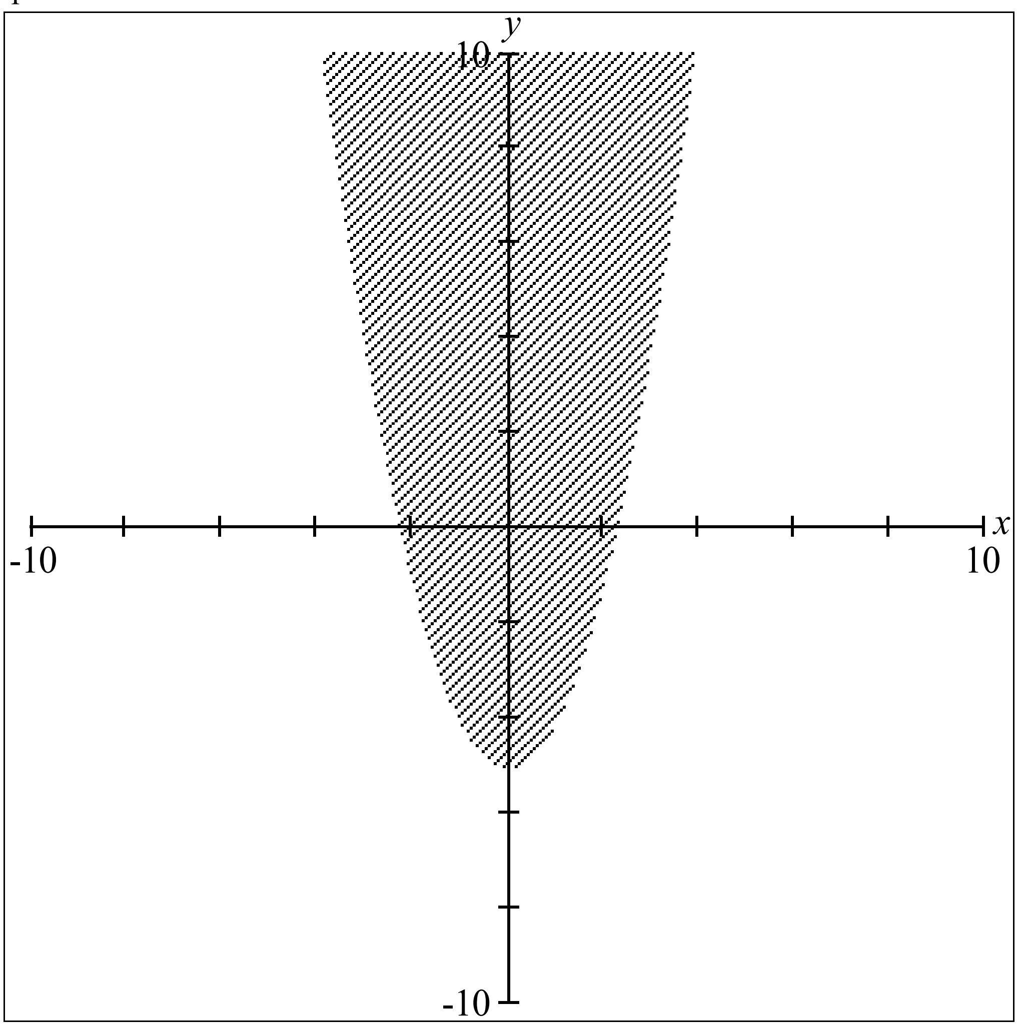


Fig.54 $y-x^2+5=|y-x^2+5|$

Therefore, to graph an inequality of the form: $a<b$, simply enter: $b-a = |a-b|$. (Note that technically you are including the edge in the graph, although this will make no difference in most inequality graphs.)

Other Lessons

Analytic Geometry

A Nice Problem

A student recently shared the following problem with me:

Given the two points A(1,4), B(5,1) and the line L: y = 2x-1, determine a point P, on L, such that the angle ∡APB is a right angle.

Although intended, I gather, as an exercise in applying the property of reciprocal opposites of the slopes of perpendiculars, the problem has three attractive attributes:

It can be solved in a variety of ways
It can profitably utilize technology
It is easily extended.

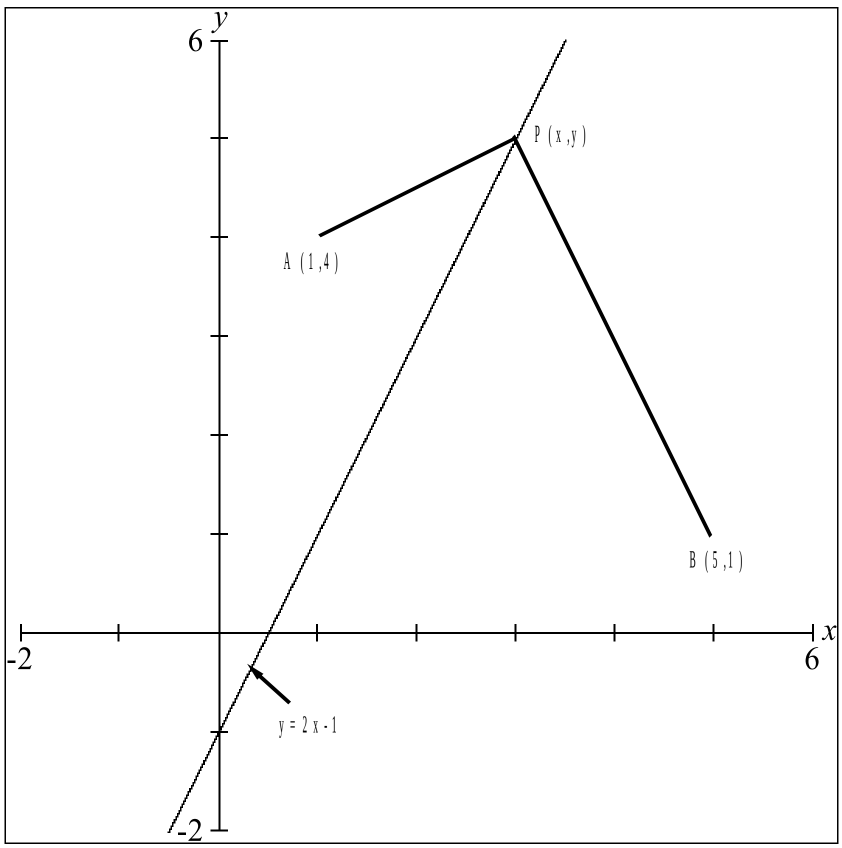


Fig.55 Two points forming a right angle at P

Following are three approaches the student might follow:

First method: note that if ∡P is a right angle, then AP ⊥ BP, and their slopes are reciprocal opposites. Therefore we may enter the following two-constraint relation, in order to plot the solution(s):

I: $y = 2x - 1$ since P(x,y) is on L

II: $\frac{y-4}{x-1} = \frac{-(x-5)}{y-1}$ since the slopes are reciprocal opposites.

Second method: note that if ∡P is a right angle then △APB is a right angle triangle, and Pythagoras' theorem applies. We thus enter the two constraints:

I: $y = 2x - 1$ since P(x,y) is on L

II': $(y-1)^2 + (x-5)^2 + (y-4)^2 + (x-1)^2 = (4-1)^2 + (1-5)^2$ since Pythagoras applies. $(PB^2 + PA^2 = AB^2)$

Third method: the mid-point M(3,2.5) of hypotenuse AB is equidistant from vertices A, B and P

I: $y = 2x - 1$ since P(x,y) is on L

II": $(x-3)^2 + (y-2.5)^2 = (5-3)^2 + (1-2.5)^2$ since MP = MB

All three methods produce the plots of two points on L : $P_1(3,5)$ and $P_2(1,1)$ which are the two solutions. This implies that all three second constraints are actually equivalent. Verification of this (by algebraic analysis) is left as an exercise. The three methods use constructs from the curriculum: slope property of perpendiculars, Pythagoras' theorem and the distance and mid-point formulæ. The third method relies on the geometric property of an inscribed angle from a diameter of a circle.

The role of the technology? The user is spared the algebraic exercise of solving a 2×2 system (but must be able to *algebraically define the situation completely.*)

And how can the problem be extended?

The third method can be expressed as a synthetic geometry construction exercise: "Given 2 points A and B and a line separating them, construct point P on the line such that ∡APB is a right angle."

What if the two points were on the same side of the line - would this alter the problem in any way -would there still be two solution points in all cases?

What if we were to change the angle from 90° to 60°? Are there any restrictions on angle choice? Which trig identities might apply?

The preceding ideas are not intended to be an exhaustive study of the problem - there may well be further ideas that can be fruitfully employed.

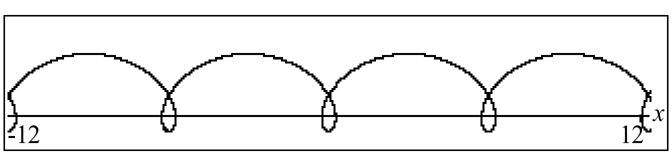
But, all in all, not a bad problem.

Curves with Names

Enrichment – Student Exercise

Following are a number of equations which define named curves. You may find further information on the history of these curves in your school library math section – possibly in a mathematics dictionary. Or on the web, there is the Famous Curves Index[4]. Some of the name choices will be obvious once you see the graph. (note that relations defined with r and or θ are polar.)

1. **Cardiod**: $r=a(1-\cos\theta)$ the locus of point P on a circle of radius a which rolls about a fixed circle of radius a.
2. **Catenary**: $y=(a/2)(e^{x/a}+e^{-x/a})=a\cosh(x/a)$ the shape of a hanging chain.
3. **Cissoid of Diocles**: $y^2(2a-x)=x^3$ This graph may be used to determine the length of the side of a cube which has twice the volume of a given cube.
4. **Conchoid**: $(x-a)^2(x^2+y^2)=b^2x^2$, or $x^2y^2=(a^2-y^2)(b+y)^2$
5. **Cruciform**: $x^2y^2-a^2x^2-a^2y^2=0$
6. **Epicycloid**: $x=(a+b)\cos T-b\cos[(a+b)T/b]$; $y=(a+b)\sin T-b\sin[(a+b)T/b]$ (parametric)
The locus of a point on a circle of radius b as it rolls on the exterior of a circle of radius a. If a=b then it is a cardiod.
7. **Folium of DesCartes**: $x^3+y^3=3axy$
8. **Hypocycloid**: $x=(a-b)\cos T+b\cos[(a-b)T/b]$; $y=(a-b)\sin T-b\sin[(a-b)T/b]$ As epicycloid, but the circle of radius b rolls on the interior of the circle of radius a.
9. **Lemniscate**: $(x^2+y^2)^2=a^2(x^2-y^2)$
10. **Limaçon**: $r=a\cos\theta+b$
11. **Lituus**: $r^2=a/\theta$ $(r>0)$
12. **Ovals of Cassini**: $r^4+a^4-2a^2r^2\cos2\theta=b^4$ The locus of a point whose distances from 2 fixed points 2a apart multiply to b^2.
13. **Rose**: $r=a\sin n\theta$ or $r=a\cos n\theta$ will have n petals if n is odd and 2n petals if n is even.
14. **Spirals**: **Logarithmic** spiral: $\log r=a\theta$
Spiral of **Fermat**: $r^2=a\theta$
Spiral of **Archimedes**: $r=a\theta$
Involute of a circle: $x=a(\cos T+T\sin T)$; $y=a(\sin T-T\cos T)$ The locus of the end point of a taught string as it is unwound from a circular spool.
15. **Strophoid**: $y^2=x^2(a+x)(a-x)$
16. **Tractrix**: $r=a\dfrac{\log(a\pm\sqrt{a^2-y^2})}{y\pm\sqrt{a^2-y^2}}$ The locus of the end of a string whose other end is pulled along a line.
17. **Trisectrix** (of **Maclaurin**) $x^3+xy^2+ay^2-3ax^2=0$
18. **Trochoid**: $x=aT-b\sin T$; $y=a-b\cos T$ If b>0, called a **prolate cycloid**; if b<0, called a **curtate cycloid**. The locus of a point of distance b from the center of a circle of radius a as the circle rolls along the x-axis.

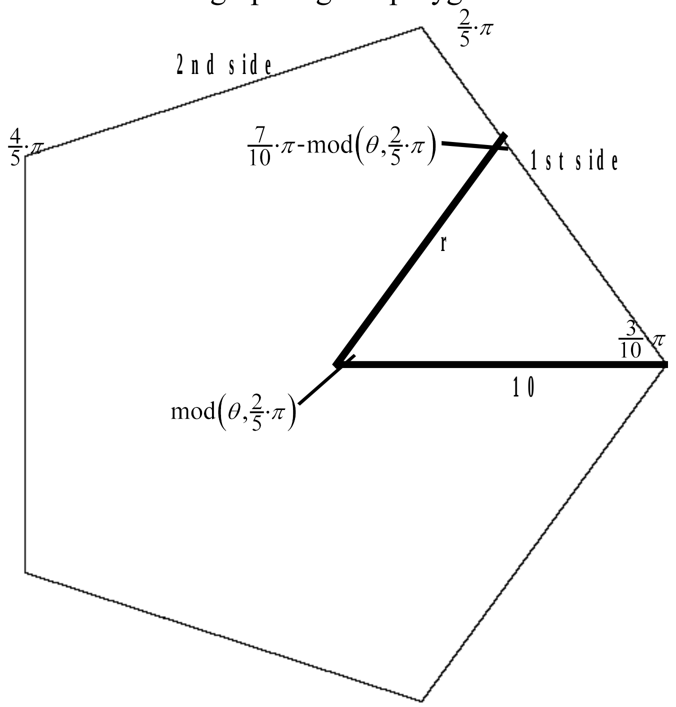


Prolate Cycloid

4 http://www-history.mcs.st-and.ac.uk/Curves/Curves.html

Plotting Regular Polygons with GrafEq

GrafEq is capable of graphing many curves - due to the variety of functions available. We use the modulo function to graph regular polygons.



$$\frac{r}{\sin .3\pi} = \frac{10}{\sin(.7\pi - \mathrm{mod}(\theta, .4\pi))}$$

$$0 < \theta < 2\pi$$

Fig.56 A Pentagon[5]

Let us consider the case of the pentagon: using polar coordinates, for a pentagon of radius 10 with the 'first' vertex at (0, 10) - the remaining vertices will have position angles of 2/5π, 4/5π, 6/5π and 8/5π. To plot the pentagon, we need to know the relation between any included point's position angle (θ) and its distance from the origin (r). If we examine a point on the 'first' side (see Fig.56) we can, using the sin law, determine the following equation: r/sin(3/10)π = 10/sin(7/10π - θ mod2/5π). When θ reaches 2/5π as we proceed to the 'second' side, θ will be reset back to 0 because of the modulo function. We can generalize the equation for any polygon:

$$\frac{r}{\sin(\pi/2 - \pi/n)} = \frac{b}{\sin(\pi/2 + \pi/n - \mathrm{mod}(\theta, \frac{2\pi}{n}))}$$

where n is the number of sides and b is the radius of the polygon.

As the number of sides increases, the polygon will become more 'circular' - let us examine the general equation as n becomes larger and larger: 2π/n will become smaller and smaller, as will π/n and θ mod(2π/n). The general equation will approach r/sin(π/2) = b/sin(π/2), which simplifies to r=b, the equation of the circle with radius b.

5 The second constraint: $0 \le \theta \le 2\pi$ assists GrafEq by limiting the angle range.

Primes?

A graphing program may seem a strange refuge for prime numbers. Nevertheless, with its ability to plot 1-dimensional relations, we are able to plot prime numbers with GrafEq.

We require an algebraic definition of prime numbers utilizing the functions available within GrafEq. The item that fits the bill is the GCD operator. Why is 7 a prime, but not 8? Because between 2 and 6 inclusive, there are *no* factors of 7 but between 2 and 7 inclusive there *is* a factor of 8.

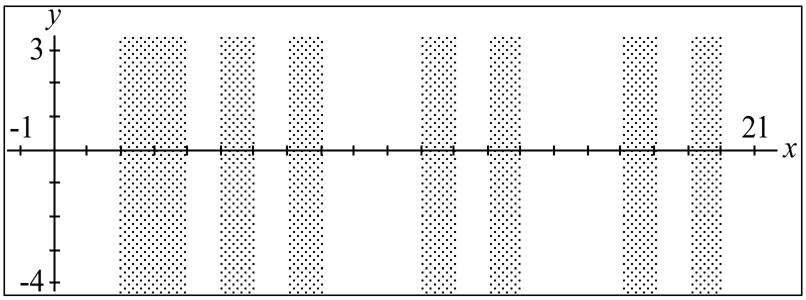


Fig.57 $\gcd(\lfloor x\rfloor, (\lfloor x\rfloor - 1)!) = 1$; $2\le x\le 20$

The above relation using gcd and the floor and factorial functions produces strips corresponding to 2,3,5,7,11,13,17 and 19.

Note that due to the magnitude of the factorial, we cannot go much beyond 20.

Pythagorean Doublets

Students of geometry are soon made aware of Pythagorean triplets: three integers such that the square of the third equals the sum of the squares of the first two. Examples are: 3-4-5 and 5-12-13.

Such triplets often apply to right triangle problems where it is desirable to have sides of integral lengths. Although there is a formula to generate such triplets (x^2-y^2,2xy,x^2+y^2) it is nevertheless interesting to note that GrafEq can generate the first two members of such triplets as is illustrated below.

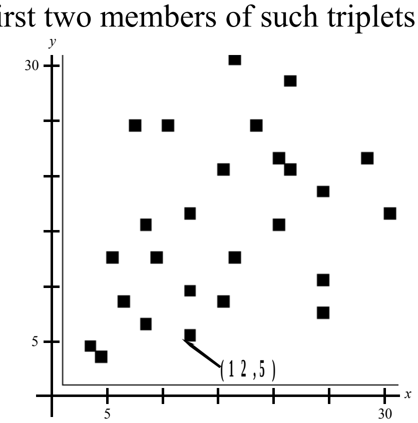


Fig.58 $\lfloor x\rfloor^2 + \lfloor y\rfloor^2 = \lfloor k\rfloor^2; 1 < k < 30\sqrt{2}; x \ge 1; y \ge 1$

Note that the four-constraint relation above does appear to plot Pythagorean doublets at the lower left corners of the squares. There appear no solutions involving a 1 or 2. Can you prove/disprove that the legs of a Pythagorean triplet cannot be 1 or 2?

Using GrafEq for Relations with a single Variable

Although we usually use a (non-graphing) calculator to simply do calculations, we can conceive of it as a device that solves equations. Equations of the form: x= f(a), where the single variable is isolated on one side, and the other side consists of constants and operations. Thus, we can use the calculator to solve x-5=2x+6.2 by re-expressing the equation as x=-5-6.2, and then entering the right side into the calculator.

GrafEq is capable of solving such relations directly; that is, without isolating the variable. This capability is most valuable for implicitly defined relations, in which the variable cannot be readily isolated. For example, the following relation: xsinx=2 yields the following graph (over the domain -2 to 10).

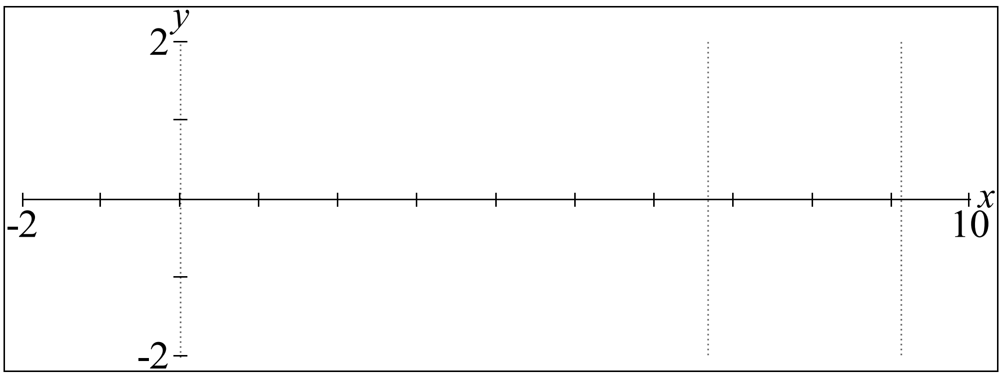


Fig.59 xsinx = 2

There appear to be three solutions: in 1-point mode they can be estimated to be: 0, 6.7 and 9.1. If we use the information mode and turn on show work, for relation #1, all three vertical lines blink. This means that GrafEq has not confirmed that any of these three are valid. We can, however, eliminate 0 by examination, since 0sin0 ≠2. And we can test the other two possibilities with a scientific calculator. For more precise solutions, we can zoom in.

We can also solve inequalities in one variable. Consider $x^4+x^3-3x^2-x<-2$. The GrafEq viewport is as in Fig.60 below.

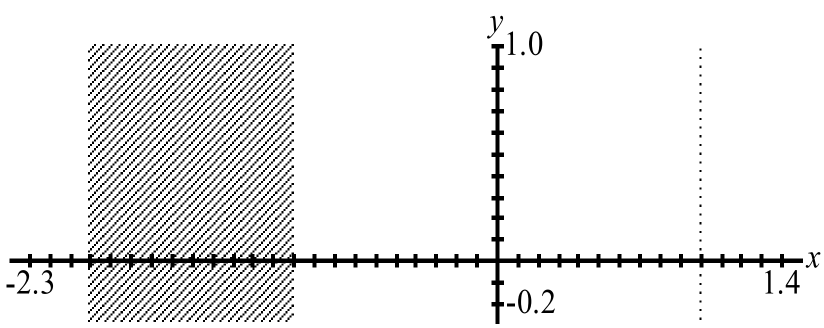


Fig.60 $x^4+x^3-3x^2-x<-2$

The graphed solutions appear to be: -2<x<-1 or x=1. Caution: the student's attention to verifying the computer display will reveal that 1 is not a solution of the inequality. (Although replacing the '<' with '≤' will bring 1 into the set of solutions.)

Reach For the Stars!

We have all at one time or another drawn free hand the five-pointed star called a "pentacle". Can it be graphed by plotting some equation?

Our first effort is to recall the equation of a point on a circle rolling inside another circle (hypocycloid). We require the smaller circle to complete two full rotations as it travels around the enclosing circle once. This yields the 2-constraint equation:

x = 6cosT + 4cos1.5T , and y = 6sinT -4sin1.5T - whose graph is shown below.

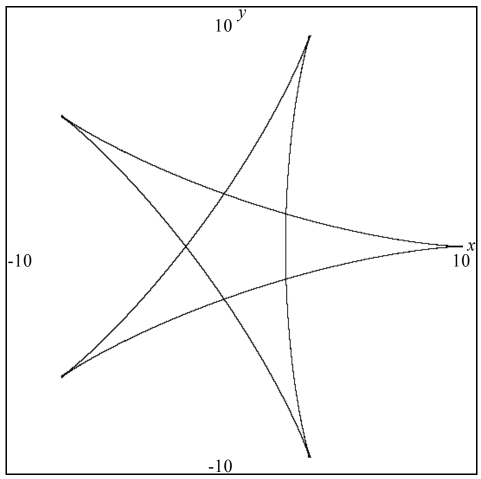


Fig.61

This is a pretty good 'first shot' at the problem - can we improve upon it by 'straightening' the sides? The answer is yes. The graph and its equation are shown below:

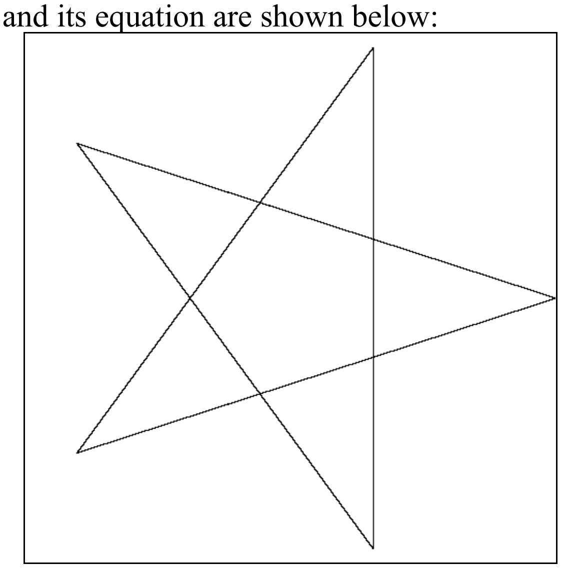


Fig.62 $r/\sin(.1\pi) = 10/\sin(.9\pi - \theta \bmod .8\pi)$

Exercise: Can you produce the equation of a 7- pointed star ('heptacle'); n-pointed star, where n is odd? Use GrafEq to check your answer.

We can produce an interesting curve if we replace r with r-2:

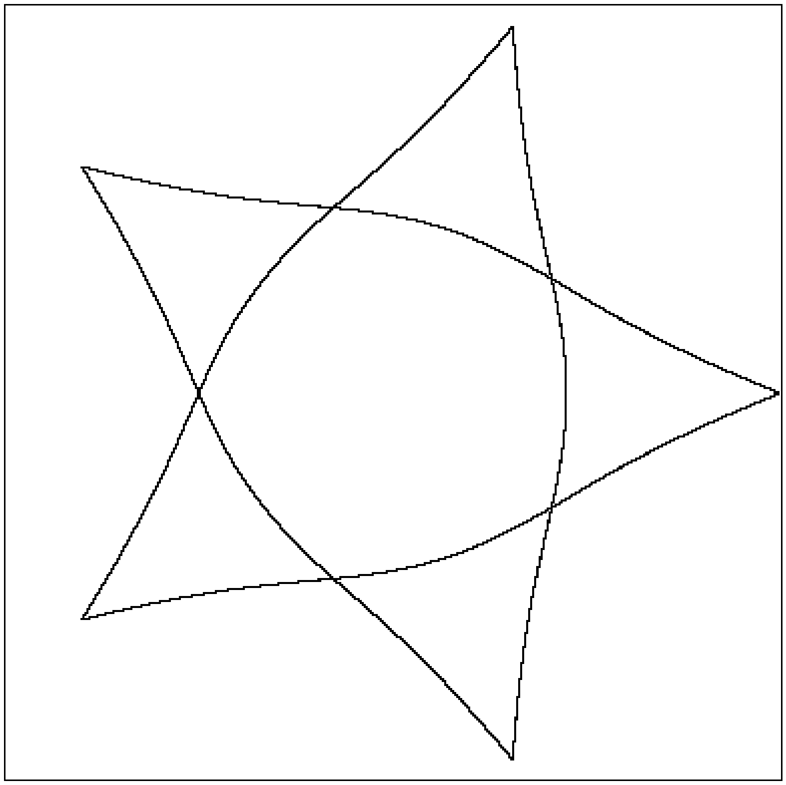


Fig.63 $(r-2)/\sin(.1\pi) = 8/\sin(.9\pi-\text{mod}(\theta,.8\pi))$

Note: GrafEq V2 produces polar relations rather slowly - GrafEq V1 will produce polar graphs more quickly. Also, in GrafEq V2, the user is advised to add an additional constraint: (e.g. $0<\theta<4\pi$) to expedite plotting.

Statistics

The Normal Distribution

Student Exercise

As you are well aware by now, we usually use tables to determine the probability of a normal random variable X. For instance, given a normal distribution with mean 70 and standard deviation 10, if you wish to determine P(60< X< 90) you will note that the region under the curve is from 1 standard deviation left of the mean to 2 standard deviations to the right of the mean. You can then use the standard normal curve tables to calculate the area: .3413 (left side) plus .4772 (right side) = .8185 The normal distribution may be defined as follows:

$$y = \frac{1}{\sqrt{2\pi}\,\delta} e^{-.5\left(\frac{x-\mu}{\delta}\right)^2}$$

You can use GrafEq to plot the normal curve. Therefore you will define the normal curve as below. In the Create View window - turn 'Preserve aspect ratio' off and specify bounds as in Fig.64.

$$y = \frac{1}{\sqrt{2\pi}\,\delta} e^{-.5\left(\frac{x-\mu}{\delta}\right)^2}$$
$$\delta = 10$$
$$\mu = 70$$

You can enter the above three-constraint relation using the 'easy buttons' (Greek) to enter the mean μ and the standard deviation δ. The curve appears as below:

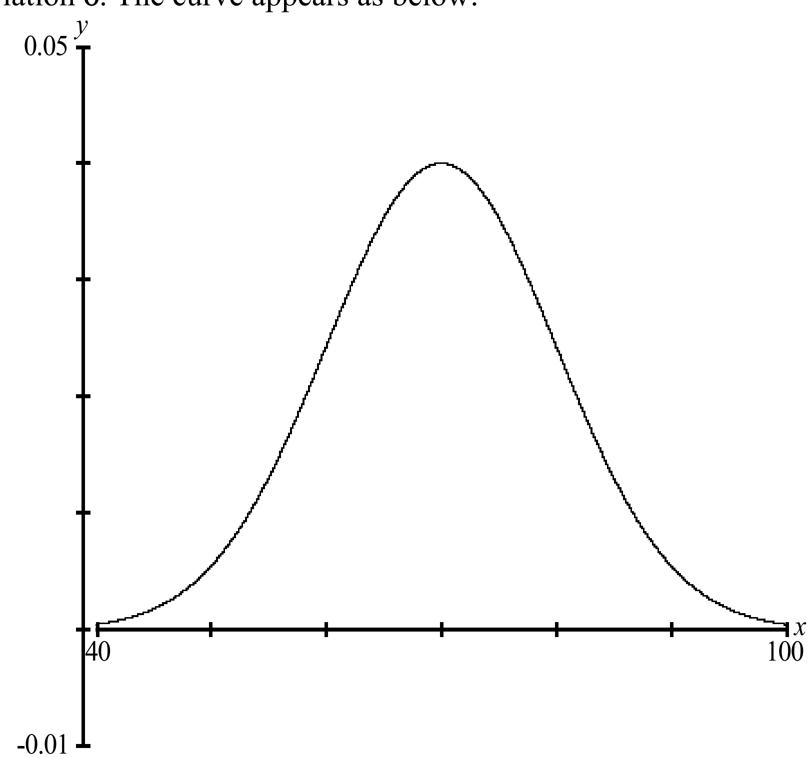


Fig.64 the Normal Curve

Vertical ticks may be added to display standard distribution intervals by the following custom ticks:

$x=\mu+\delta k$
$\mu=70$
$\delta=10$
$k=\{\pm 1, \pm 2, 0\}$

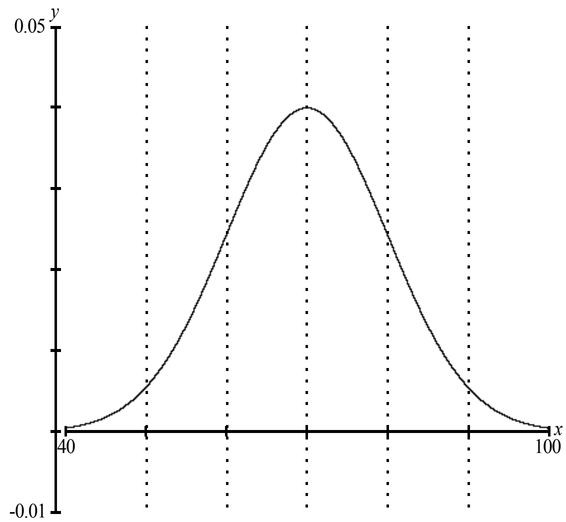


Fig.65 Custom Ticks for sd intervals

Note that custom tick labels have been set to *none*

You can actually use the graph to estimate probabilities as follows: For instance, to determine the probability that x is 60, we can *interpret* that to mean the probability that x is between 59.5 and 60.5. A reasonable estimate is the height of the curve at x=60. You can use the 1-point mode (after a few *zoom-ins*) to determine the y-coordinate of the point with x-coordinate 60.000 to be about .024

So, $P(x=60) \approx .024$

Estimate the following probabilities:

A. $P(X=80)$ = ___________

B. $P(X=35)$ = ___________

C. $P(62.5<X<65.5)$ = __________

Trigonometry

Sin by Series

Student exercise - Enrichment

How Close is Close Enough?

(Approximating the Sine function with a polynomial)

GrafEq can be used to give us some insights which are not nearly so readily attained in an environment limited to pencil and paper and numerical calculator.

To illustrate the above notion, let us examine a process for approximating a trig function with a polynomial: specifically – "How many terms of the Maclaurin series : $x - x^3/3! + x^5/5! - x^7/7!$ .. are required to provide an accurate value for sin(x)?" It is obvious from the form of the series that it alternates - ultimately each successive term brings us closer to the limiting value, on alternate sides.

We can better understand the situation if we graph y=sin x and $y = x - x^3/3! + x^5/5!$ simultaneously. (see Fig.66) We note that the three terms of the series approximate the sine curve quite accurately between -2 and 2. If we add the fourth term ($-x^7/7!$) and graph again (see Fig.67) we note that the range of accuracy has been broadened. It is also clear from the graph that the accuracy depends primarily on the value of x chosen: the four-term polynomial is quite inaccurate for any values outside the domain $-3 \le x \le 3$.

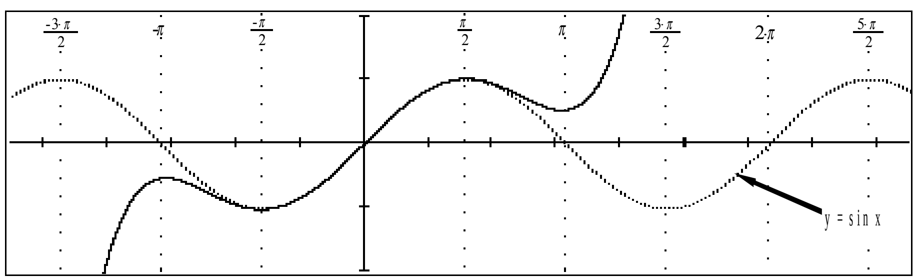


Fig.66 $y = x - x^3/3! + x^5/5!$ (dark curve)

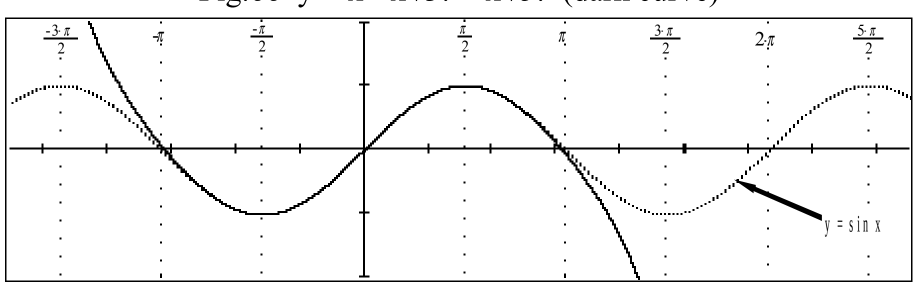


Fig.67 $y = x - x^3/3! + x^5/5! - x^7/7!$

Since the sine function is periodic, we realize that it is unnecessary to accurately approximate the function over its complete domain - a single period[6] of high accuracy will suffice.

Let us examine the actual discrepancy of the 5-term polynomial by graphing $y = x - x^3/3! + x^5/5! - x^7/7! + x^9/9! - \sin x$. (see Fig.68) We see that the difference is very insignificant in the interval $(-\pi/2, \pi/2)$ If we add another term ($-x^{11}/11!$) and regraph we see that the domain of accuracy has been further broadened to (-2,2). (see Fig.69)

6 We can use the identity $\sin x = \sin(x \pm n2\pi)$ to evaluate outside the domain $(-\pi, \pi)$.

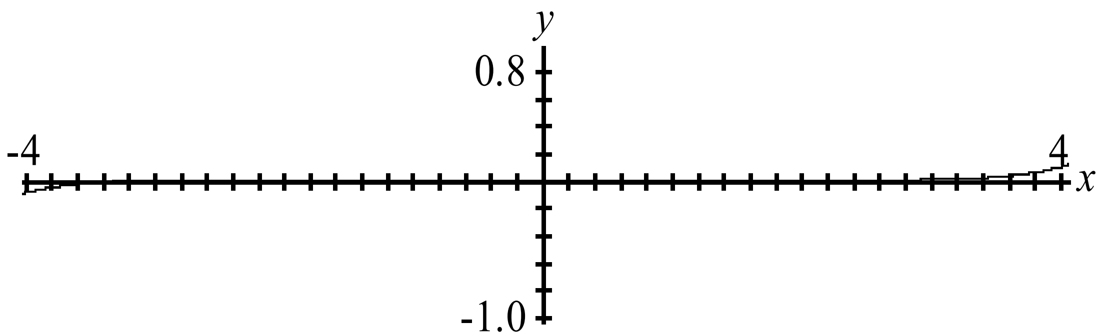


Fig.68 $y=x-x^3/3!+x^5/5!-x^7/7!+x^9/9!-\sin(x)$

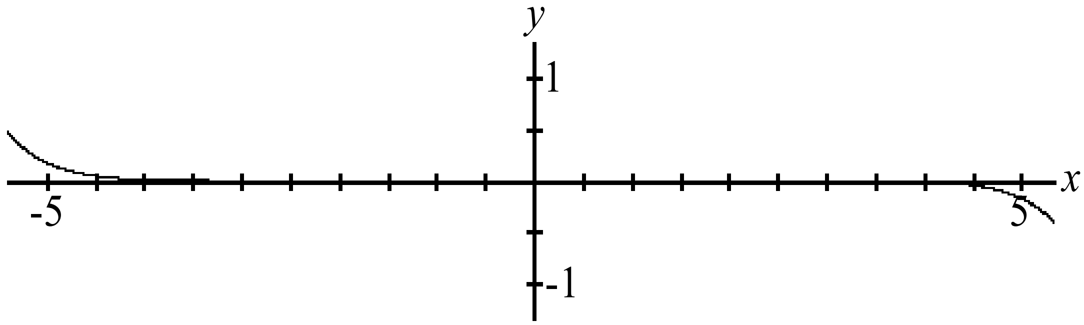


Fig.69 $y=x-x^3/3!+x^5/5!-x^7/7!+x^9/9!-x^{11}/11! -\sin(x)$

To achieve great accuracy over the domain of one period ($-\pi$, π) will require the addition of some extra terms, but at this stage we realize that the sine curve is completely determined by only one-quarter of its period - that is, the curve from 0 to $\pi/2$ can be used to create the entire curve. Therefore it is only necessary to minimize the error at $\pi/2$[7]. Fig.70 clearly illustrates that the degree 11 approximation is out by an amount significantly less than 0.00001 at $\pi/2$.

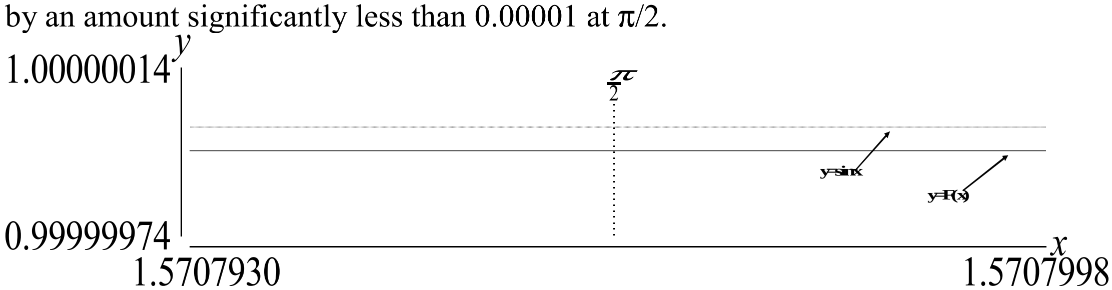


Fig.70 $y=\sin x$ and $y=x-x^3/3!+x^5/5!-x^7/7!+x^9/9!-x^{11}/11! =F(x)$ near $x=\pi/2$

We should remind ourselves at this stage that what the computer displays as the sine curve is itself an approximation - possibly the very polynomial we have been studying! You will have noted that the title How close is close enough? is rather vague. The best answer is perhaps: "How close do you need?"

Further exercises:

1. Express the following using arguments between 0 and $\pi/2$: (a) $\sin(2\pi+1)$ (b) sin 10 (c) $\sin -\pi$
   Confirm your answers with a calculator.
2. $y = \sin(x)$ and $y = x - x^3/3! + x^5/5!$ have one simultaneous solution at (0,0). Are there any others? Hint: Use the 'zoom' feature to study the graph of $y = x - x^3/3! + x^5/5! - \sin(x)$ very close to the x-axis.
3. The author implicitly assumes that the error increases with distance from the origin. Can you confirm that this is true (or false)?

---

7 If, for instance, we require accuracy to the nearest thousandth, we note that an approximation using n terms of the series is out by no more than the $n+1^{th}$ term, since the terms alternate between positive and negative. Therefore if we wish to approximate $\sin(\pi/2)$ to the nearest thousandth we seek n such that $(\pi/2)^n/n! \leq .0005$. The smallest odd value of n that suffices is 9, so we use four terms of the polynomial.

The Sine Function

Student Assignment

$$y = A \sin(Bx - C) + D$$

We shall use GrafEq to determine the graphical significance of the constants A, B, C and D. You are already aware of the geometric definitions of amplitude, period and phase shift.

1. Launch GrafEq by double-clicking on its icon. Click (twice) to dispose of the title screen.

2. Enter the following initial relation:

$$y = A\cdot\sin(B\cdot x - C) + D$$

$$A = 1$$
$$B = 1$$
$$C = 0$$
$$D = 0$$

3. In the Create view window, set y-bounds at 6 and -6. Leave the x-bounds at ±10. Enter <t> to access the ticks mode and turn ticks on. The basic curve (y=sinx) will appear thus:

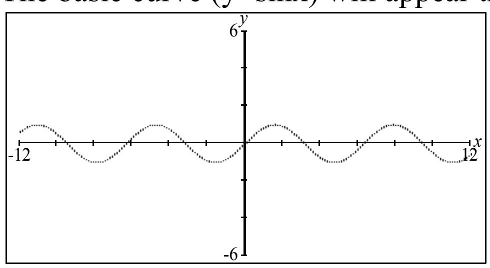


Fig.71 y=sinx

To increase the tick density, press on the third tick button. Since the sine curve has a period of 2π, it will be desirable to view the domain in terms of multiples of π. Therefore,

4. Under File select New custom ticks – preformatted multiples of π/2 except 0 The view window will appear as below:

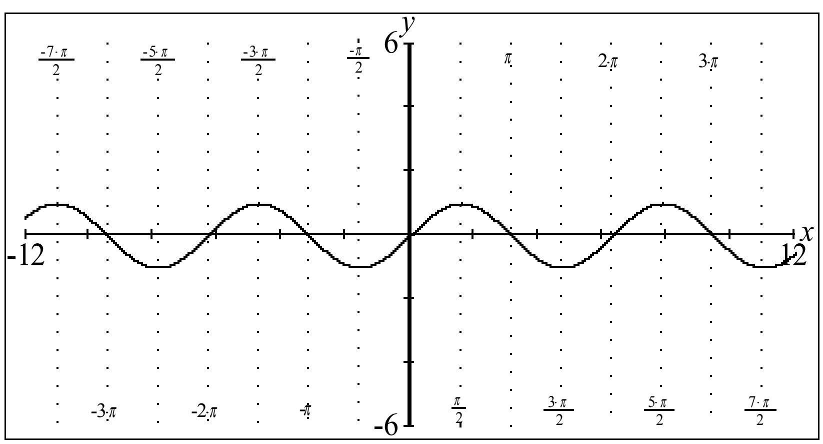


Fig.72 y=sinx with vertical ticks

5.Note that the period is 2 π and the amplitude is 1. Enter these values in the first line of the table.

6. Since we will wish to compare the future graphs to the 'basic' curve: y=sinx, we will keep this

relation and its graph, and create a *second relation* for various substitutions in A, B, C, and D. Therefore to create a second relation,

7. Ensure that the relation#1 window is active (foremost with darkened title bar) by either clicking on the relation#1 window or by selecting 'Relation #1' under Graph. Under Edit, select Copy.

8. Under Graph, select New Relation. Under Edit, select Paste. You can now carry on, with *all subsequent changes of A, B, C, and D to be made in the Relation#2 window.* Complete lines 2 and 3 of the table by successively specifying the values 1/2 and 2 for B in the relation #2 window. Note that all three curves pass through (0,0) and have the same amplitude, but different periods. How does the value of B effect the period? Confirm your answer by specifying a value of 4 for B and graphing the resultant equation. Finally, specify a value of -2 for B. Note that although the graph has been 'flipped' (horizontally) its period is the same as line 3 (ie π). We can summarize the above information by saying the period will be $2\pi / |B|$.

9. Deactivate relation#2 by clicking on the B=-2 field. Edit that field to read: B=1. The fundamental relation#1 (y=sinx) will remain in the view window. Note that the graph passes through (0,0). Now, reset C in the relation#2 window to $\pi/4$ and re-activate. Note that the curve has shifted $\pi/4$ to the right. Enter $\pi/4$ under phase shift. Alter C to $-\pi/4$ and regraph and complete line 9 in the table. For line 10, specify a C value of 1 and re-graph. Note that the graph has shifted ____ to the right. Enter 1 under phase shift in line 10.

10. Deactivate the current graph (relation#2) and edit the appropriate fields A, B, C, D to have values of 1, 2, 1 and 0. Enter the resultant phase shift in line 11. Enter the line 12 values for the constants to confirm that the phase shift is .25 to the left. (ie -1/4) The general rule is contained in line 13. Deactivate Relation #2

11.Enter the line 14 values in the appropriate fields and regraph. Note that the amplitude (of y=sinx) is ____ Reset A to 2 and regraph to complete line 15. Complete line 15 and line 16 in the same way. The conclusions are expressed in line 17. Before completing line 18, anticipate all the missing values and confirm by graphing. Complete line 19 to determine the effect of D. (Note that the resultant values are the same as line 18 but ____
____)

TABLE

y = Asin(Bx-C)+D

	A	B	C	D	equation	amplitude	period	phase shift
1.	1	1	0	0	y = sin x	_______	_____	XXXXXX
2.	1	1/2	0	0	y = sin .5x	_______	_____	XXXXXX
3.	1	2	0	0	y = sin 2x	_______	_____	XXXXXX
4.	1	4	0	0	y = sin 4x	_______	_____	XXXXXX
5.	1	-2	0	0	y = sin(-2x)	_______	_____	XXXXXX
6.	A	B	C	D	y=Asin(Bx-C)+D	XXX	2 /|B|	XXXXXX
7.	1	1	0	0	y = sin x	1	2π	0
8.	1	1	/4	0	y = sin (x -π/4)	_______	_____	π/4 right
9.	1	1	- /4	0	y = sin (x +π/4)	_______	_____	_________
10.	1	1	1	0	y = sin (x - 1)	_______	_____	_________
11.	1	2	1	0	y = sin(2x - 1)	_______	_____	_________
12.	1	-4	1	0	y = sin(-4x - 1)	1	π/2	_________
13.	A	B	C	D	y=Asin(Bx-C)+D	XXX	2π/|B|	C/B right if pos-left if neg
14.	1	1	0	0	y = sin x	_______	_____	_________
15.	2	1	0	0	y = 2sin x	_______	2π	0
16.	-3	1	0	0	y = -3sin x	_______	_____	_________
17.	A	B	C	D	y=Asin(Bx-C)+D	|A|	2π/|B|	C/B
18.	2	4	π/2	0	y=2sin(4x-π/2)	_______	_____	_________
19.	2	4	π/2	2	y=2sin(4x-π/2)+2	_______	_____	_________

The effect of D is ____

____

Extras

Periodic Function - Variable frequency

GrafEq gracefully handles functions with rapid oscillations. A simple example of a "periodic" function which demonstrates this is y=sin(100/x)

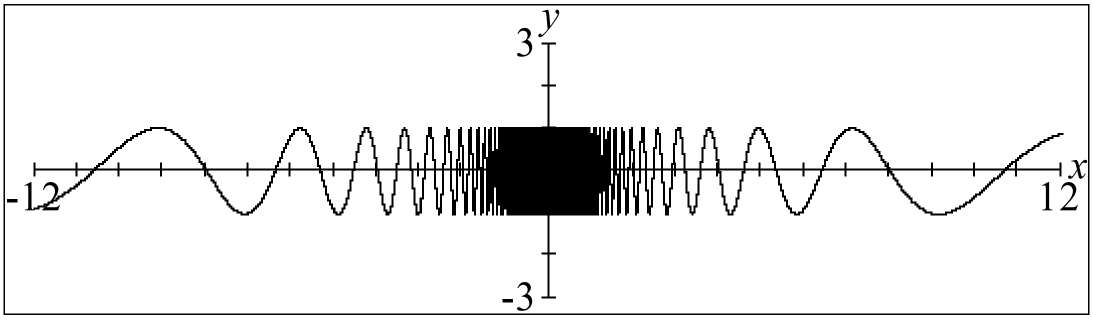


Fig.73 y=sin(100/x)

Look at the upper and lower bounds of the plot. The plot oscillates between -1 and +1, more and more rapidly as the y-axis is approached. Looking at the equation, this is to be expected. What is interesting about this example?

It demonstrates an advantage of the technique GrafEq uses for plotting. Many other graphing packages plot functions by computing f(x) for various values of x and then "connect the dots", as shown in the following picture:

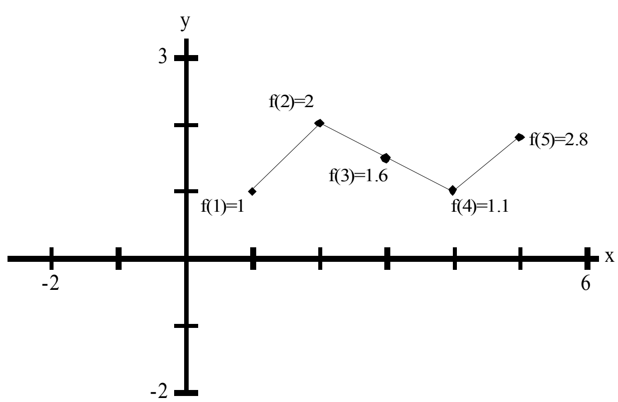
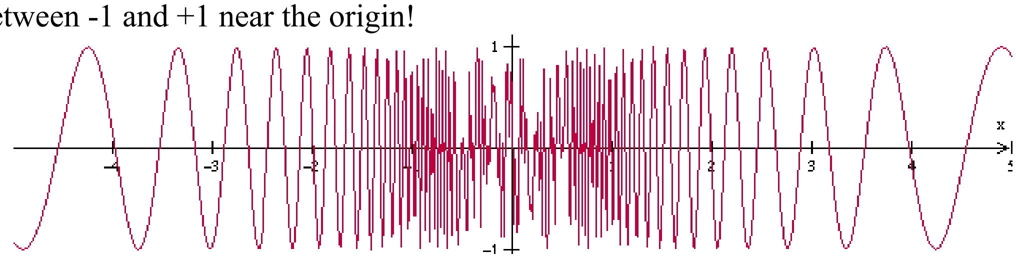


Fig.74

Using such an approach, the function f(x) = sin(100/x) must be "sampled" (evaluated) very densely near the origin (it need not be densely sampled far away from the origin) to produce an accurate plot. Try plotting the above function on another graphing program - the produced plot may differ significantly from that produced by GrafEq - quite often the produced plot would lead you to believe the function does not oscillate between -1 and +1 near the origin!

Example of mis-plot:



Periodic Function -High frequency

GrafEq gracefully handles functions with rapid oscillations. A simple example of a periodic function which demonstrates this is y=sinkx with a large value for k. The plot GrafEq produces below is the intersection of the two constraints y=sinkx and k=5.

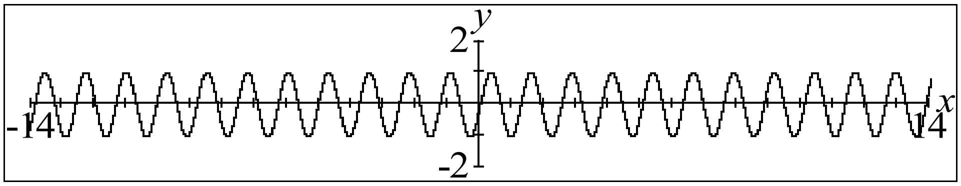


Fig.75 y=sin5x

The plot is as expected, but what happens if we increase the constant k? Below is a plot for k=20. To reproduce, simply click on the constraint k=5 and edit the right hand side.

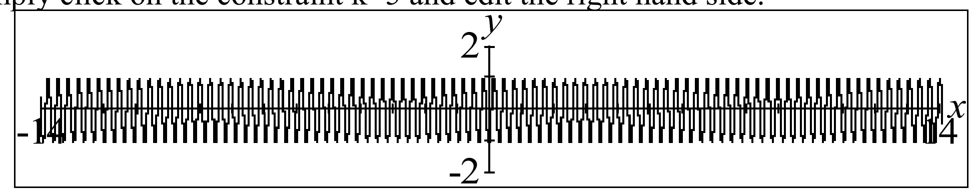


Fig.76 y=sin20x

Still, nothing too unexpected. The plot is getting a little thick, let's try k=500.

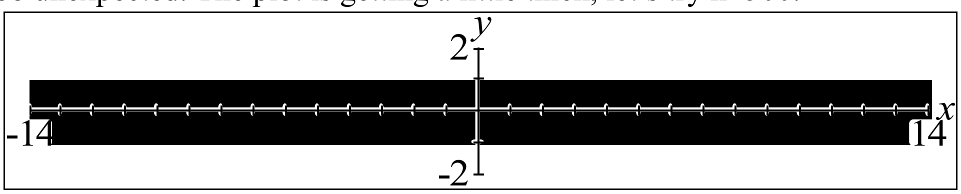


Fig.77 y=500x

Is the plot correct? Having such a large value of k causes the sine wave to oscillate between -1 and +1 very rapidly - the wave reaches it's maximum (and minimum) around 20×500/2π times in the above plot. In fact, the length of a period of the sine wave is shorter than the width of a pixel, which causes the plot to be solid (remember that GrafEq's goal is to darken a pixel if and only if there is a solution inside the pixel). The plot is correct in that GrafEq has attained its goal for this plot.

Unfortunately, the plot would lead one to believe that the function is solid; perhaps GrafEq's goal is too simple. There is a limit to how well any plotting technique can work - there are only a finite number of pixels on any computer screen, hence any single plot (by any given program) can only convey a finite amount of information.

Further things to consider - y=sinkx can be thought of as a three dimensional equation in x, y, k space. Try to visualize the three-dimensional graph - k=5 is a plane in three dimensions so the above plots are actually planar slices through the three dimensional graph, with the slicing plane at various positions. Consider other forms for the second constraint (for example $k=1/x^2$).

Plotting a Function with a Sharp Spike

Functions with sharp spikes can cause problems for many plotting programs. GrafEq gracefully handles functions with arbitrarily sharp spikes.

The plot GrafEq produces is displayed below as Fig.78. It suggests that the function is essentially the straight line y=x with a vertical asymptote near x=3.

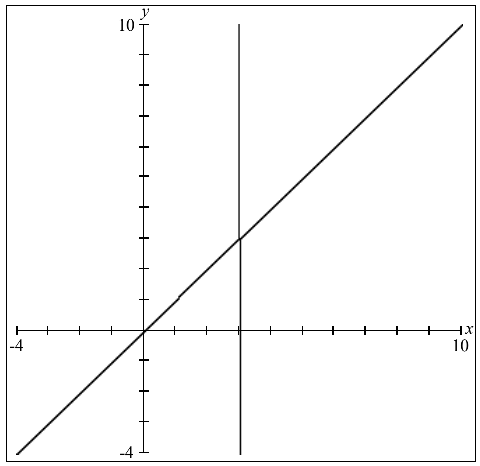


Fig.78 $y = \frac{x(x-3.0001)}{x-3}$

Let's analyze the equation of Fig.78:

$y = \frac{x(x-3.0001)}{x-3} = x\frac{x-3-.0001}{x-3} = x\left(1-\frac{0.0001}{x-3}\right)$ Clearly the function can be viewed as the product of f_1=x and $f_2 = 1-\frac{0.0001}{x-3}$ The vertical asymptote of f_2 as well as of the plotted function, is x=3 as expected. More insight is gained by noticing that f_2 is approximately equal to 1 except when near the asymptote. Obviously this is true theoretically (from the definition of vertical asymptote) but it also true in a more practical sense: even when x is a pixel width away from the asymptote, f_2 is so close to 1 as to cause y to fall in the same pixel row as it would for the equation y=x. As a result, many plotting programs will plot a straight line for the above equation. Such programs can be "pushed" simply by bringing the constants in the numerator and denominator very close to one another. (Making the spike skinnier makes it less likely for such plotting programs to try a value of x which causes y to differ noticeably from y=x)

The second plot (Fig.79) is of the equation $y = \sin\frac{x(x-3.0001)}{x-3}$ By including f_2 in equations little spikes can be added where desired. Several spikes may be added to functions by including f_2 several times, with different numerators and denominators.

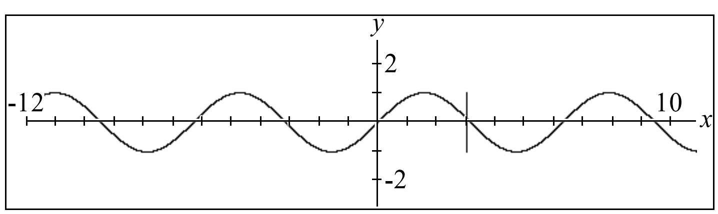


Fig.79 $y = \sin\frac{x(x-3.0001)}{x-3}$

Locus with GrafEq

Consider the following locus problem:

Given: A point F(a,b) and a line *l* not containing F. For any point Q on *l* determine the locus of point P which is the intersection of the perpendicular bisector of QF and the line through Q perpendicular to *l*.

Solution using GrafEq: with no loss of generality, let *l* be the y-axis and therefore (see illustration below) point Q will have coordinates (0,y). Point P has coordinates (x,y).Point M has coordinates (a/2,(y+b)/2). The slope of QF is (b-y)/a and the slope of its perpendicular is a/(y-b). The coordinates of F (a and b) may be selected arbitrarily (a≠0).

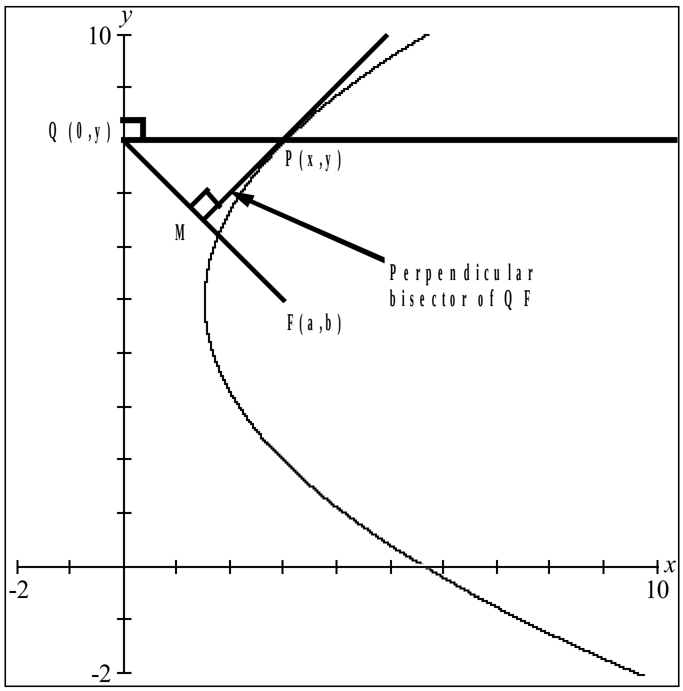


Fig.80

We may now determine the constraints as follows:

$$y - \frac{b+y}{2} = \frac{a}{y-b}\left(x - \frac{a}{2}\right)$$

$$a = 3$$

$$b = 5$$

The first constraint is the equation (point-slope form) of the line through M with slope a/(y-b). The other two constraints are the arbitrary values of F's coordinates. The graph is displayed as a parabola with its directrix on the y-axis and F is the focus.

It can be left as an exercise for the student to convert constraint 1 into the standard form.

This example also implies a convenient method to construct a parabola's envelope.

Discrete Functions with GrafEq

GrafEq can plot many discrete functions with ease using equations. As an example, we will plot the function $y=(-1.4)^x$ where x is integral. A relation is discrete if its graph is a collection of isolated points.

The following plot will appear shortly after entering the following three constraints:

$$y = (-1.4)^x$$

$$x = \lfloor x \rfloor$$

$$y = \begin{cases} |y| \; if \; \mathrm{mod}(x,2) = 0 \\ -|y| \; if \; \mathrm{mod}(x,2) = 1 \end{cases}$$

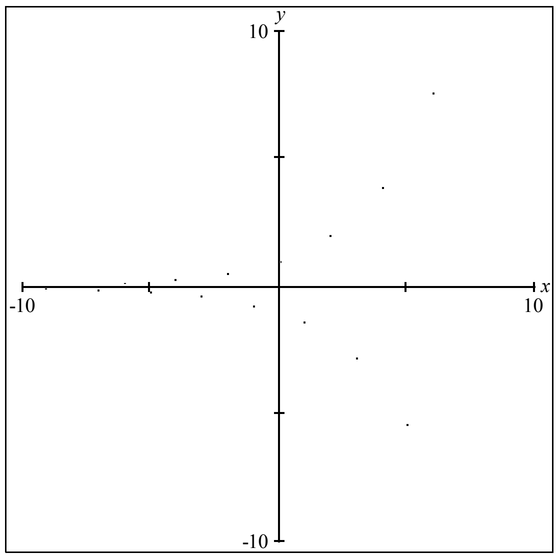


Fig.81

Plotting discrete relations shows off one of the advantages of the plotting technique GrafEq uses. The plot is quickly produced and the plot satisfies GrafEq's ultimate plotting goal (a pixel is darkened if and only if a solution is found within the region the pixel represents).

Unfortunately, on high-resolution devices, the individual pixels can be difficult to discern. One solution is to select a black background with white relation.

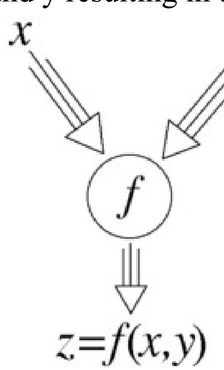
Neighbours

This short document explains some of the internal workings of GrafEq. In addition to scalar floating-point ("regular") arithmetic GrafEq also employs interval floating-point arithmetic. It is assumed the reader is familiar with scalar real arithmetic (arithmetic of the form 3+7=10 or π+2π =3π etc...) Only two further things need to be understood.

Floating Point

First, floating point. A floating-point number is a real number that may be expressed in a finite number of digits (in some base). For example, in base ten, 12.71 is a floating point number. Floating point numbers can be expressed in scientific notation, our previous example being $1.271 \times 10^1=12.71$. The term "floating point" refers to the decimal point being determined by the exponent. Computers, being rather slow and simple (currently!), are commonly programmed to use floating point to keep things simple and fast (rather than using real numbers). To further simplify things, computer floating-point numbers usually have a fixed number of digits, and limit the exponent to a small range of integers.

How does the use of floating point affect GrafEq? For simplicity assume GrafEq uses base ten floating-point numbers with four digits. The diagram below illustrates a function f being applied to two arguments x and y resulting in the value z=f(x,y) being computed.



For a simple, concrete example assume the function is addition, x=12.45 and y=1.006. What is the value of z? If z were a real number it would be 13.456. Unfortunately, there is no floating point number equal to 13.456 (since all numbers in our example floating point system have only four digits - 13.456 has five digits). What to do? The usual solution is to round up so that z=13.46. Using the above floating point system adding 12.45 to 1.006 produces the result 13.46. This is somewhat shocking - floating point number systems do not follow all of the field axioms[8] - ((x+y)-x)-y = ((12.45+1.006)-12.45)-1.006 = -0.006 using the above floating point number system. When using real numbers the above obviously produces 0. Addition is not commutative!

Considering all of the problems of floating point numbers you may initially find it somewhat amazing that programs using floating-point work at all. The key thing to know is that most computer systems use floating point numbers with much finer granularity than the above system. For example, many programs use a base two floating point system with more than 50 digits of (base two) accuracy. It can take many operations for the difference between floating point and real numbers to become considerable[9].

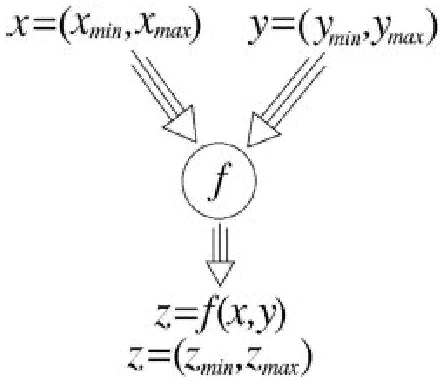
8 An axiom may be thought of as a basic truth. A field is composed of a number system and the basic operations + and x. Field axioms are basic truths about the operations + and x.

9 The difference between the real number result and the floating-point result is sometimes referred to as "round-off error".

Interval Arithmetic

GrafEq uses interval arithmetic during plotting, as well as at other times. An example of an interval would be $x=[2.3,\pi]$ which includes every number from 2.3 to π, including 2.3 and π. Two types of intervals will concern us initially: real intervals and floating point intervals. A real interval consists of real numbers while a floating-point interval consists of floating point numbers.

Arithmetic may be performed on intervals just as it may on scalars. The sum of the interval $[2.3,\pi]$ and $[-4,3]$ is the interval $[-1.7,3+\pi]$. To add two intervals simply add the upper and lower bounds together. As with floating point numbers, not all of the field axioms hold for interval arithmetic. For example, intervals do not have additive inverses. Adding the interval [1,2] to [-2,-1] results in the interval [-1,1]. In general, to perform operation f on intervals $x=(x_{min},x_{max}) = [x_{min},x_{max}]$ and $y=(y_{min},y_{max})$ perform the operation f on every pair of numbers $X[x_{min},x_{max}]$, $Y[y_{min},y_{max}]$ and pick the smallest interval $z=(z_{min},z_{max})$ which includes every number f(X,Y). The resultant interval must include every possible result possible with the source intervals[10].



GrafEq uses floating-point intervals. The combination of floating point and intervals add additional problems not present when using either individually, as well as removing some problems that are present using either individually. The point of this small note is to explain what the terms nearest neighbor and best neighbor mean. Consider adding the two floating-point intervals [10.00,12.00] and [1.009,2.001]. 10.00 + 1.009 = 11.009 while 12.00 + 2.001 = 14.001 using real arithmetic. The closest floating-point numbers (nearest neighbor) are 11.01 and 14.00, which lead to the interval R,=[11.01,14.00]. Unfortunately 10.00+1.009=11.009 which is not in the interval R. Choosing the floating point numbers 11.00 and 14.01 (best neighbor) result in the interval Q=[11.00,14.01]. The interval Q is a superset of the real interval [11.009,14.001] so that Q is a better choice for the resultant floating point interval. Best neighbor arithmetic satisfies the goal of arithmetic interval at the expense of enlarging the resultant interval. GrafEq v2.00 uses nearest neighbor arithmetic although best neighbor may be implemented in a future version.

Intensity Fields

With a little work GrafEq can plot intensity fields. What are intensity fields? It is a plot where the intensity at a pixel conveys the value at that point. This short note will explain some basic methods to get GrafEq to plot various types of intensity fields. Version 2.00 of GrafEq does not directly support this type of plotting. In fact GrafEq produces ‘digital’ plots - each pixel is either on or off. How can we get around this problem? (We are interested in doing intensity plots where there are more than two intensities).

Many common output devices are limited to digital output. Newspapers are invariably printed with

10 Also notes when f(X,Y) may be undefined.

‘digital’ output devices (in that at each point ink is either placed or not placed). Yet newspapers commonly have reproductions of photographs which appear not to be limited to two different intensities. The techniques of using binary output devices to produce output with more than two intensities is called “halftoning”. Computer based halftoning is commonly called “digital halftoning”. So, in this note I will present some simple digital halftoning techniques.

Rectangular Grid

A very simple method of halftoning is to place solid squares of varying sizes on a square grid. The size of the square placed on the grid represents the intensity throughout that grid position.

The grid positions can be labeled with integer coordinates. Example positions on the grid are (1,3) and (-2,7). The (horizontal or vertical) distance between neighboring grid positions will be a fixed distance k. Therefore grid position (i,j) will be “near” Cartesian coordinate (ki,kj).

We will be plotting the function *f*(x,y). The method presented may be applied to any function. The minimum and maximum value of *f* over the plotting region has to be determined. Call these values *m* and *M* - *m* corresponds to “white” while *M* corresponds to “black”. Changing these values alters the contrast and brightness of the produced plot.

Consider the following set of constraints:

$$\frac{f - m}{M - m} > \max\left(kx - \lfloor kx \rfloor, ky - \lfloor ky \rfloor\right)$$

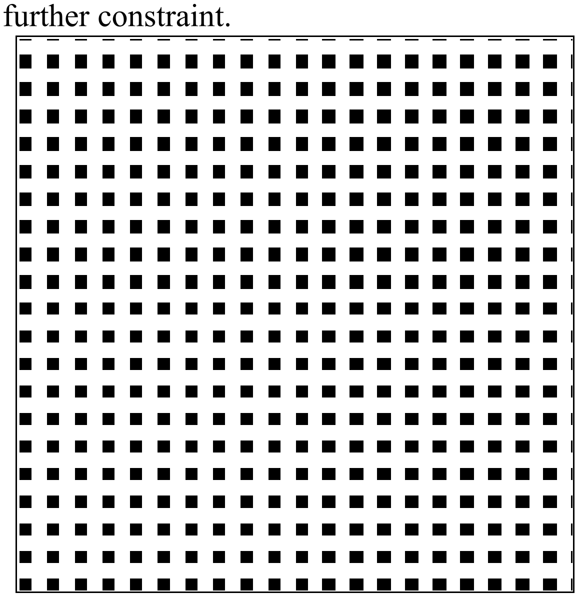
$$k = 1$$

$$m = 0$$

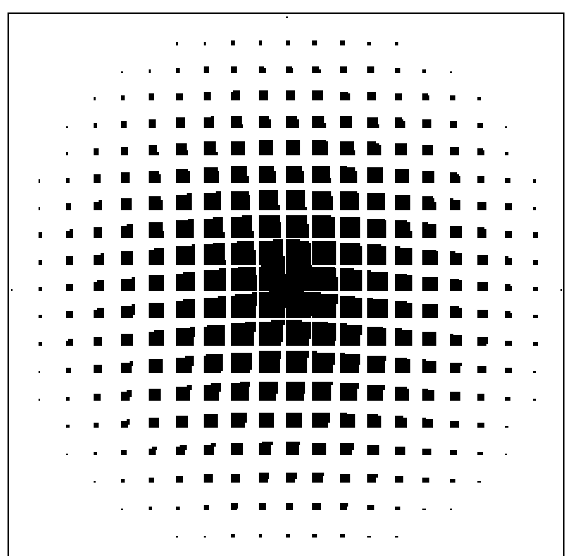
$$M = 10$$

The point (x,y) will satisfy the above constraints if the value of *f*(x,y) is over the threshold $\max\left(kx - \lfloor kx \rfloor, ky - \lfloor ky \rfloor\right)$. The left part of the first constraint simply normalizes *f*(x,y) so that it may be compared with the thresholding function $\max\left(kx - \lfloor kx \rfloor, ky - \lfloor ky \rfloor\right)$. It is the thresholding function that specifies that solid squares will be placed on the grid.

To produce the plots below enter the above set of constraints as well as the function (such as *f*=4) as a further constraint.



$f=4$

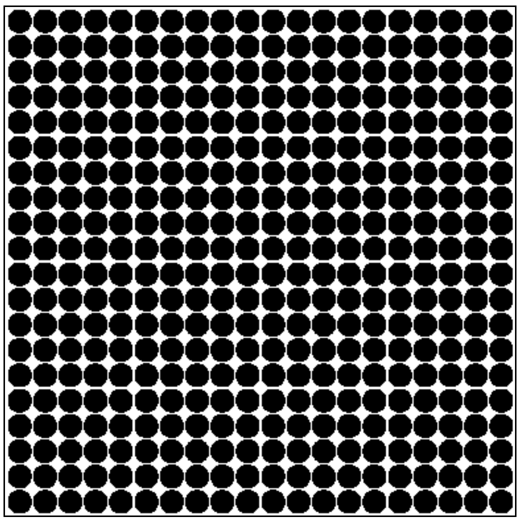


$f=10 - \sqrt{x^2 + y^2}$

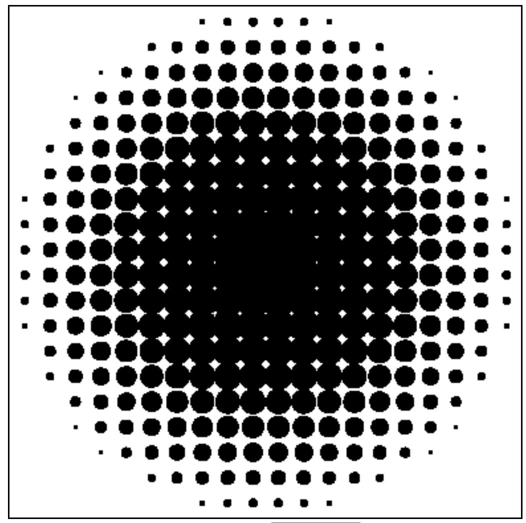
A major drawback of this technique is that the human visual system can pick up horizontal and vertical lines quite easily. Since the squares are aligned with the grid, vertical and horizontal lines are quite appafrent. Using solid disks rather than squares on the grid can downplay these lines. To accomplish this the thresholding function must be changed. The following will produce disks:

$$2\left[\left(kx-\lfloor kx\rfloor-\tfrac{1}{2}\right)^2+\left(ky-\lfloor ky\rfloor-\tfrac{1}{2}\right)^2\right]$$

A poor quality of the thresholding functions presented so far is that the area covered by the spot in a grid position is not proportional to the value of the function f. This may be remedied by careful modification of the thesholding function.



$f=4$



$f=10-\sqrt{x^2+y^2}$

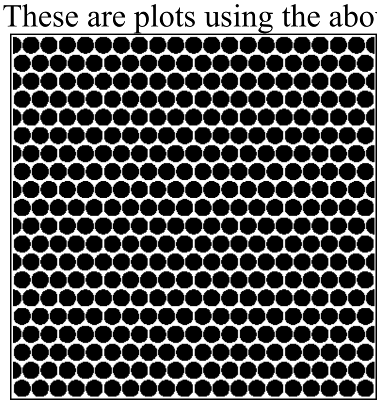
Hexagonal grid

Rather than placing the disks on a rectangular grid, they may be placed on a hexagonal grid. 45^0 lines are not seen as easily as verticals. A thresholding function that places disks on a hexagonal grid is:

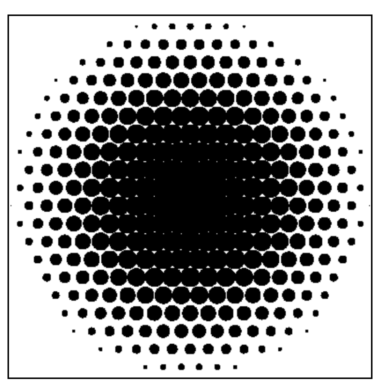
$$2\left[\left(k(x+o)-\lfloor k(x+o)\rfloor-\tfrac{1}{2}\right)^2+\left(ky-\lfloor ky\rfloor-\tfrac{1}{2}\right)^2\right]$$

$$o=\frac{\left\lfloor \frac{ky}{2}\right\rfloor-\tfrac{1}{2}\lfloor ky\rfloor}{k}$$

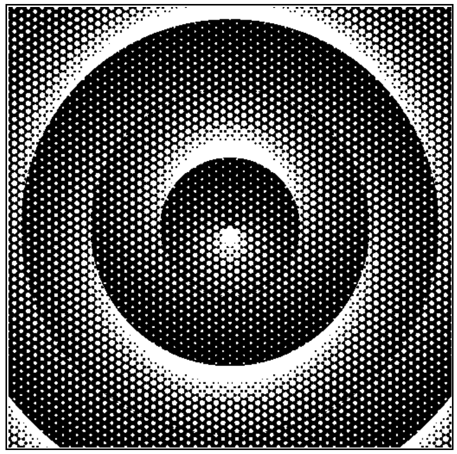
These are plots using the above:



$f=4$



$f=10-\sqrt{x^2+y^2}$



$$\frac{2}{\pi}\,\mathrm{mod}(a,\pi) + \sin\left(b + \pi\left\lfloor\frac{r}{\pi}\right\rfloor\right) - 1 > \cos 20x + \cos 10\left(x - \sqrt{3}y\right) + \cos 10\left(x + \sqrt{3}y\right); a = \sqrt{x^2 + y^2}; b = angle(x, y)$$