# THE MORLEY TRISECTOR THEOREM

CLETUS O. OAKLEY AND JUSTINE C. BAKER

The great algebraic geometer Frank Morley (1860–1937) came to this country from England to teach at Haverford College in 1887. In 1900 he moved on to Johns Hopkins to head their department of mathematics. (For *in memoriam* résumés of his life and works see [33], [92].) In 1900 his brilliant paper "On the Metric Geometry of the Plane *n*-line" appeared in the first issue of the *American*

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Cletus O. Oakley received his Ph.D. at the University of Illinois under R. D. Carmichael. He has taught at the University of Texas, University of Puerto Rico, Dillard and Brown Universities, and at Haverford College, and is now retired. His research interests are in nonlinear equations. He has written some fifteen books, seven with coauthor C. B. Allendoerfer. For two years he was a Fulbright scholar at the University of Western Australia and University of Tasmania.

Justine C. Baker is a Ph.D. student at the University of Pennsylvania. She has taught both mathematics and science at the secondary level. Her research interests are in both pure mathematics and mathematics education. She has published two short books concerning computers.—*Editors*

*Mathematical Society Translations* [**78**]. In it he proved several very general theorems about the behavior of $n$-lines in the plane and their characteristic constants. In passing, it is interesting to note that in this important memoir not once is use made of the now customary format: *Theorem*...*Proof.*

But among these unannounced theorems there is a very, very special case of one which has intrigued mathematicians for the past three-quarters of a century. It is now simply known as **Morley's trisector theorem.**

*The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle.*

It is one of the most astonishing and totally unexpected theorems in mathematics and, jewel that it is, for sheer beauty it has few rivals. The simplest case involves only the interior angles. In [**13**], [**81**], [**82**], you will find Morley's thoughts on how the theorem arises quite naturally from his own contributions to what has now become known as the theory of Clifford chains. They are coded CC in the references.

There have appeared in print many proofs of this theorem. We believe references to most of them will be found in our verified list. With popularization and proliferation of proofs, it is understandable that some uncertainties and errors of fact concerning the origin, statement, and earliest printed proofs have crept into the literature. Perhaps this note can set much of the record straight. To begin with, it is *Frank* Morley, not *John*, as stated in [**31**], [**44**].

Morley, of course, was well aware of the unique characteristics of his theorem and its ramifications. Indeed, his theory accounted for all 18 cases of Morley equilateral triangles, but it pleased him to indicate that he had not bothered to make a big song and dance about it since it was only a small part of his general theory. And so he never enunciated, in print, just the simple theorem, nor did he ever publish a direct verification of it.

However, he had not been slow in communicating it to his friends, such as Richmond at Cambridge and Whittaker at Edinburgh, and by 1904 it had become public. See Morley's letter to G. Loria in [**71**].

The earliest printed statements of the theorem we have found are those of E. J. Ebden, who apparently was so taken with the problem that he introduced it, simultaneously, in the British Isles and on the continent. In 1908 it appeared in *The Educational Times*, London [**42**], [**101**], as problem 16381, and in *Mathesis*, Brussels [**36**], as problem 1655—in both instances without the benefit of Morley's name. But this is not surprising, because for some years the theorem seemingly floated around in search of an author; and as late as 1913 Taylor and Marr read a paper on the subject before the Edinburgh Mathematical Society without knowing the authorship of the theorem. An acknowledgment is in their paper [**108**], which, by the way, was the first to give the complete solution.

The solution to Ebden's problem 16381 is given by Satyanarayana in [**101**] and to his problem 1655 by Delahaye and H. Lez in [**36**]. The elegant proof of the latter consists in finding the length of a side of the Morley triangle. Let the given triangle be $ABC$ with interior angles $A=3\alpha$, $B=3\beta$ and $C=3\gamma$. Let $O$ be the center and $OC=r$ be the radius of the circumcircle and let the Morley triangle be $DEF$ (see Fig. 1). In our notation they found $EF=8r\sin\alpha\sin\beta\sin\gamma$, which, by symmetry, proves that $DEF$ is equilateral. This seems to be the first occurrence of this formula, although Kaven [**61**] states that Hofmann [**54**] was the first to use it.

To the best of our knowledge, the next earliest proof is in [**83**] by M. T. Naraniengar, *Mathematical Questions and Solutions*, from *The Educational Times*, with many papers and solutions in addition to those published in *The Educational Times*, London, New Series, 15 (1909) 47. We have spelled out the reference in detail here because confusion does arise: this paper, apparently, did not appear in *The Educational Times*. It occurs in the *Mathematical Questions and Solutions*, which is often referred to as the "Reprints" from *The Educational Times* (misspelled "Repruits" in [**61**], causing more confusion).

By 1920 the problem had aroused so much interest that it was set in the St. John's group of Entrance Scholarships.

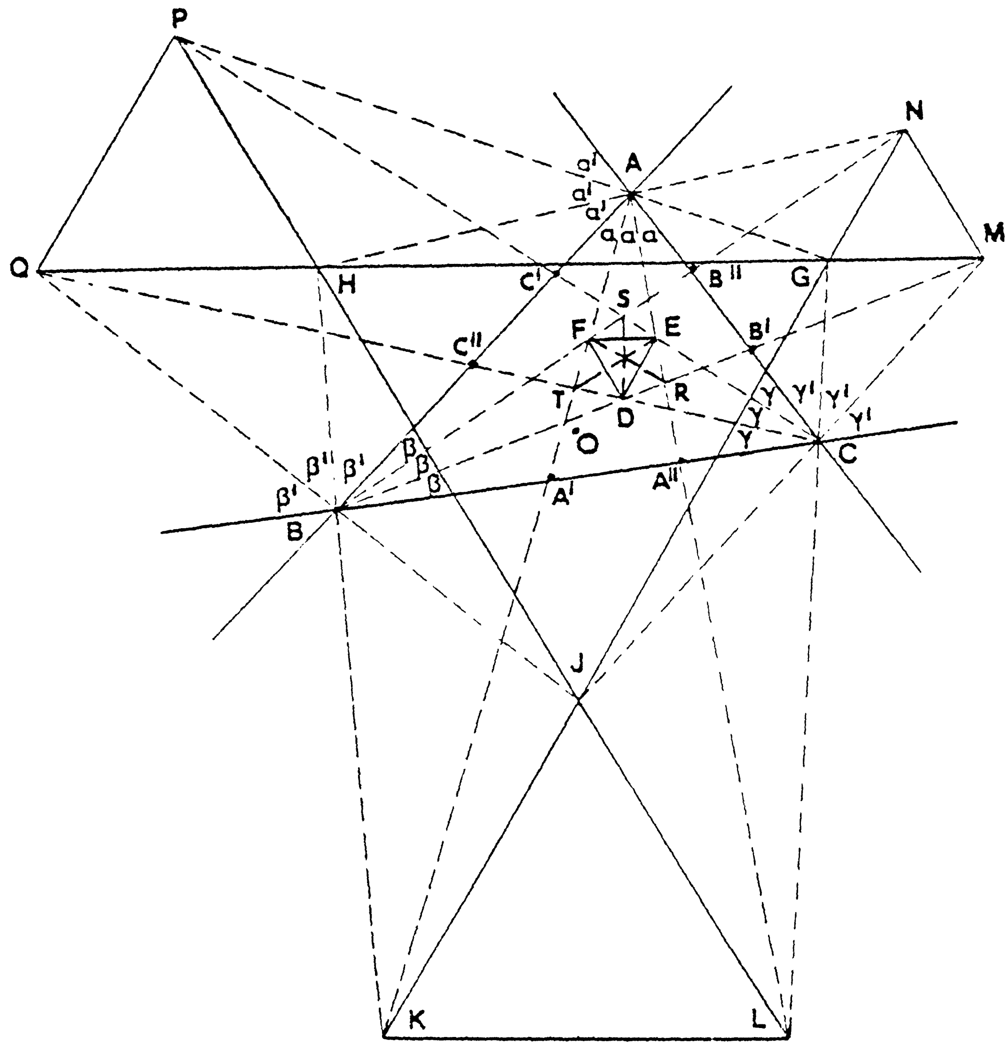


FIG. 1.

Specializations of Morley's general theory of 1900 hold (see Fig. 1) for (i) interior angles, yielding triangle *DEF*, (ii) exterior angles, yielding triangle *GHJ*, and (iii) a mixture of two exterior angles and one interior angle resulting in three Morley triangles, *JKL*, *GMN*, and *PQH*. Both Satyanarayana and Delahaye–Lez treat all three cases as do [**44**], [**69**], [**85**].

But there are still further generalizations since each angle $A, B, C$ can be trisected in three distinct ways by using $A$, $A+2\pi$, $A+4\pi$, and similarly for $B$ and $C$. When this is done, there arise 27 triangles, 18 of which are equilateral. These 18 include the ones previously noted and constitute the complete solution (CS). The figure for the complete solution is complicated. See [**38**], [**55**], [**108**]. All Morley triangles have parallel sides, and the length of a side of each is of the form

$$8r\sin(\alpha+\theta)\sin(\beta+\phi)\sin(\gamma+\psi)$$

where each of $\theta$, $\phi$, $\psi$ is some particular arrangement of 0, $\pi/3$, $\pi/6$ [**71**].

We have included reference materials, coded R, *related* material, because, as you might expect, Morley's theorem has connections with many notable point-line-plane-circle-polygon configurations bearing such names as Apollonius, Brianchon, Ceva, Desargues, Feuerbach, Hesse, Lemoine, Menelaus, Pascal, Ptolemy, Simson, Spieker, Steiner, etc. Indeed, some of the proofs begin with a named theorem. See, for example, [**48**], where the beginning cites Desargues and Menelaus. Again, in [**85**], Neuberg credits Ad. Mineur with a proof which first notes that hexagon $A'A''B'B''C'C''$ is Pascal and hexagon *DRESFT* is Brianchon (see Fig. 1). In the footnote of [**85**, p. 363] correct $A'C$ to read $AC'$. And make the same correction in [**74**], [**109**] where the error has been repeated, possibly through careless editing.

For the interested reader, we recommend the following papers for both the variety and ingenuity they offer: [**32**], [**48**], [**71**], [**88**], [**106**], [**108**], [**113**].

All of the proofs that we have seen use only elementary mathematics, but few of them can be said to be simple. Some polish off their proofs with a two- or three-line flourish *after* starting with as many as three lemmas, usually involving somewhat complicated trigonometric identities. By finding the length of a side of the Morley triangle, a number of papers follow the general pattern of [**36**], but none so succinctly as [**66**]. Because of its brevity we forthwith give [**66**] in its entirety (with only a change in notation to fit Figure 1).

$$AF=\frac{c\sin\beta}{\sin(\alpha+\beta)}=\frac{2r\sin\beta\sin 3\gamma}{\sin\left(\frac{\pi}{3}-\gamma\right)}=8r\sin\beta\sin\gamma\sin\left(\frac{\pi}{3}+\gamma\right).$$

Similarly

$$AE=8r\sin\beta\sin\gamma\sin\left(\frac{\pi}{3}+\beta\right)$$

But

$$\overline{EF}^2=\overline{AE}^2+\overline{AF}^2-2\,\overline{AE}\cdot\overline{AF}\cdot\cos\alpha.$$

And it follows that $EF=8r\sin\alpha\sin\beta\sin\gamma$. By symmetry this proves the theorem.

Come on now and be a good sport: fill in the trigonometry—and time yourself!

It occurred to us that some readers might be interested in having a more personal look at the man who so nonchalantly tossed off this everlasting geometric gem. Accordingly, we asked the youngest of the Morley sons, Frank V. Morley, who had worked with his father in geometry, to share with us some of his thoughts. Here are his comments.

"I was a school-boy when my father, who was almost forty years older than I was, sketched for me, free-hand, a pencilled diagram of the simplest form of the above-discussed theorem in plane geometry.

"I tested it at once with my own drawing instruments. No matter what the shape of the original triangle I started with, there in its midriff was an equilateral triangle, picked out by the trisectors. It was wizard, it was weird—and it was True!

"Always, to the eye at least, the theorem, if you drew accurately, proved itself. What caused me considerable annoyance was that I could not for a long time comprehend what purblind examiners might accept as a valid proof (*demonstratio mirabilis sane*). But before I could prove the theorem, I went on drawing diagrams, and a secondary wonder emerged: how the simplest diagram had remained a secret until my father spotted it. People had been toying with ruler and compasses and poring over the geometry of the triangle in their many generations—at least since the time of Euclid—how come nobody broke the taboo on trisecting angles—how come nobody had drawn the trisectors and *seen* the equilateral triangle inside?

"Now my father did not lack warmth for any geometrical property so simple and startling as this one. I never asked him outright the question, though it is a proper one, that Professor Oakley now asks me: namely, why at the time of discovery my father kept his cool about promoting the 'gem'—there might have been some bit of hoo-ha if he had removed the cover and sent it to the show-room as a separate static cut stone. I think the way the theorem is presented in the book *Inversive Geometry* [13] may answer the question. Attention to the detached theorem was not, for him, to interfere with the pleasure of watching his 'mobile' of cardioids and their tangents: it was the cardioids which led him to, and provided for him the most elegant proof of, the trisector theorem. Proof and theorem were pleasing in their togetherness. Isolate the theorem if you wish, but for him the cardioids in their happy behaviour were 'in beauty surpassing the Princes [*sic*] of Troy.'

"Nevertheless the ease with which simple diagrams of trisectors of a triangle's angles may be drawn certainly suggested that someone, somewhere, might have visualized and commented on the theorem before my father's contemplation of the $n$-line brought him to it. Hence, I think, my father's quiet, semi-private mentions of the theorem to expert colleagues, as occasion offered. He was not informed by any colleague he tried, in the U.S.A., or Britain, or European countries, of any prior knowledge of the existence of the 'gem.' His permission for publication of the theorem in Japan elicited no prior knowledge of it in the Far East either. I think he would have agreed that by now you could put his name to it.

"As to portraiture of that Frank Morley, author of the above-discussed theorem and others, I did the best I could some years ago in a small book called *My One Contribution to Chess* (originally published by B. W. Huebsch, New York, 1945). But copies of that, if they now exist, must be rare."

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## References

The following letter-coding of the reference numbers should be clear and, we trust, useful. They give some indication of the mathematical nature of the references.

- B. Book
- CC. Mathematics associated with Clifford chains
- CS. Complete solution (for all 18 Morley triangles)
- CV. Proof using complex variables
- G. Proof by geometry
- IP. Indirect proof
- PG. Proof by projective geometry
- PP. Proposed problem (Morley, or related)
- PPS. Proposed problem solved
- R. Related material
- T. Proof by trigonometry

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**113**T. R. J. Webster, Morley's triangle theorem, Math. Mag., 43 (1970) 209–210.

**114**CC. F. P. White, An extension of Wallace's, Miquel's and Clifford's theorems on circles, Proc. Cambridge Philos. Soc., 22 (1925) 684–687.

**115**G, T, R. Max Zacharias, Über den Zusammenhang des Morleyschen Satzes von den winkeldrittelnden Eckenlinien eines Dreiecks mit den trilinearen Verwandtschaften im Dreieck und mit einer Konfiguration $(12_4; 16_3)$ der Dreiecksgeometrie, Deutsche Mathematik, 3 (1938) 36–45.

**116**T. G. B. Zecca, Period. Mat., IV Ser., T.i. (1921) 220, Morley problem proposed by R. Marcolongo. Solved by Zecca, p. 291.

In October, 1977, we sent a preliminary copy of the above list of references to Professor H. S. M. Coxeter for his comments. When he replied, he told us that Charles W. Trigg, Professor Emeritus, Los Angeles City College, had prepared a similar list, of approximately the same length, for publication in *Eureka*, a monthly mathematics journal published by Algonquin College, Ottawa, Canada (Editor: Léo Sauvé, Algonquin College, 281 Echo Drive, Ottawa, K1S 1N3). Through Professor Coxeter's good offices, and with the informal cooperation of *Eureka* and this MONTHLY, it was agreed to combine the two lists of references in the following way. Our list of 116 coded items (above) would be published in both *Eureka* and this MONTHLY. Professor Trigg's items *not* in our list would follow in both journals so that, in effect, the complete list of references would be equivalent to one of joint authorship. The two reference lists, numbered consecutively, were printed in *Eureka*, 3, No. 10, (Dec. 1977) 281–290, along with the following Morleyana items:

1. Presenting the Morley issue of *Eureka*, p. 272.
2. On the intersections of the trisectors of the angles of a triangle, F. Morley, p. 273–275 (item [**81**] in our list).
3. Notes on Morley's proof of his theorem on angle trisectors, Dan Pedoe, pp. 276–279.
4. Robson's proof of Morley's theorem, pp. 280–281, (item [**96**] in our list).
5. An elementary geometric proof of the Morley theorem, Dan Sokolowsky, pp. 291–294. To our knowledge, this is the only paper on the Morley theorem that considers not only the interior angles $3\alpha$, etc., and the exterior angles $\pi-3\alpha$, etc., but also the reflex angles $\pi+3\alpha$, etc.
6. The beauty and truth of the Morley theorem, Leon Bankoff, pp. 294–296.

**Supplementary List of References to the Morley Theorem**

(Prepared by Charles W. Trigg, Professor Emeritus, Los Angeles City College)

**117.** Anon., Morley's Theorem, Indiana School Mathematics Journal, 10, No. 3 (1975) 1–3.