Note on Mr Tweedie's Theorem in Geometry.

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Let ABC, A′B′C′ (Fig. 4) be two triangles equiangular in the same sense. Let BC, B′C′ meet in X. Describe circles round BXB′, CXC′ to meet again in O. Then it is easy to see that the triangles BOC, COA, AOB are equiangular in the same sense to the triangles B′OC′, C′OA′, A′OB′ respectively. Hence the triangles AOA′, BOB′, COC′ are similar;

$$\therefore \quad \frac{AA'}{AO} = \frac{BB'}{BO} = \frac{CC'}{CO};$$

$\therefore$ $a \cdot AA'$, $b \cdot BB'$, $c \cdot CC'$ are proportional to $a \cdot AO$, $b \cdot BO$, $c \cdot CO$, where a, b, c are the sides of the triangle ABC.

From O draw OP, OQ, OR perpendicular to BC, CA, AB respectively.

$$\text{Then} \quad QR = AO\sin A \propto a \cdot AO,$$
$$RP = BO\sin B \propto b \cdot BO,$$
$$PQ = CO\sin C \propto c \cdot CO;$$

$\therefore$ $a \cdot AA'$, $b \cdot BB'$, $c \cdot CC'$, being proportional to $a \cdot AO$, $b \cdot BO$, $c \cdot CO$, are proportional to QR, RP, PQ.

But PQR is a triangle, unless O is on the circumcircle of ABC when PQR is the Simson line of O.

$\therefore$ $QR + RP > PQ$, with two similar inequalities, except that *one* of the inequalities becomes an equality if O is on the circumcircle of ABC.

$\therefore$ $a \cdot AA' + b \cdot BB' > c \cdot CC'$, with two similar inequalities; one of the inequalities becoming an equality when O lies on the circumcircle of ABC.

Similarily in the case of an equality O lies also on the circumcircle of A′B′C′.

For the case of equilateral triangles $a = b = c$;

$\therefore$ $AA' + BB' > CC'$, with two similar inequalities; one of the three inequalities becoming an equality when O lies on the circumcircles of ABC and A′B′C′.

It is obvious that the theorem reduces to Ptolemy's Theorem or its converse.

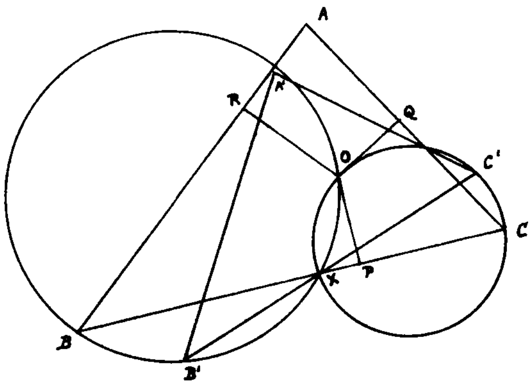

Fig. 4.