THE SUMS OF SQUARE TECHNIQUE

I. Theorem.

Consider the following inequality

$$m\sum_{cyc}a^4+n\sum_{cyc}a^2b^2+p\sum_{cyc}a^3b+g\sum_{cyc}ab^3-(m+n+p+g)\sum_{cyc}a^2bc\ge 0$$

With a,b,c *be real numbers.*

Then this inequality holds when $\begin{cases} m>0 \\ 3m(m+n)\ge p^2+pg+g^2 \end{cases}$.

Proof.

We rewrite the inequality as

$$m\left(\sum_{cyc}a^4-\sum_{cyc}a^2b^2\right)+(m+n)\left(\sum_{cyc}a^2b^2-\sum_{cyc}a^2bc\right)+p\left(\sum_{cyc}a^3b-\sum_{cyc}a^2bc\right)$$
$$+g\left(\sum_{cyc}ab^3-\sum_{cyc}a^2bc\right)\ge 0$$

Note that

$$\sum_{cyc}a^4-\sum_{cyc}a^2b^2=\frac{1}{2}\sum_{cyc}(a^2-b^2)^2$$
$$\sum_{cyc}a^3b-\sum_{cyc}a^2bc=\sum_{cyc}b^3c-\sum_{cyc}a^2bc=\sum_{cyc}bc(a^2-b^2)$$
$$=-\sum_{cyc}bc(a^2-b^2)+\frac{1}{3}(ab+bc+ca)\sum_{cyc}(a^2-b^2)=\frac{1}{3}\sum_{cyc}(a^2-b^2)(ab+ac-2bc)$$
$$\sum_{cyc}ab^3-\sum_{cyc}a^2bc=\sum_{cyc}ca^3-\sum_{cyc}ab^2c=\sum_{cyc}ca(a^2-b^2)$$
$$=\sum_{cyc}ca(a^2-b^2)-\frac{1}{3}(ab+bc+ca)\sum_{cyc}(a^2-b^2)=-\frac{1}{3}\sum_{cyc}(a^2-b^2)(ab+bc-2ca)$$

Then the inequality is equivalent to

$$\frac{m}{2}\sum_{cyc}(a^2-b^2)^2+\frac{1}{3}\sum_{cyc}(a^2-b^2)[(p-g)ab-(2p+g)bc+(p+2g)ca]$$
$$+(m+n)\left(\sum_{cyc}a^2b^2-\sum_{cyc}a^2bc\right)\ge 0$$

Moreover

$$\sum_{cyc}a^2b^2-\sum_{cyc}a^2bc=\frac{1}{6(p^2+pg+g^2)}\sum_{cyc}[(p-g)ab-(2p+g)bc+(p+2g)ca]^2$$

The inequality becomes

$$\frac{m}{2}\sum_{cyc}(a^2-b^2)^2+\frac{1}{3}\sum_{cyc}(a^2-b^2)[(p-g)ab-(2p+g)bc+(p+2g)ca]$$

$$+\frac{m+n}{6(p^2+pg+g^2)}\sum_{cyc}[(p-g)ab-(2p+g)bc+(p+2g)ca]^2\ge 0$$

$$\Leftrightarrow \frac{1}{18m}\sum_{cyc}[3m(a^2-b^2)+(p-g)ab-(2p+g)bc+(p+2g)ca]^2$$

$$+\frac{3m(m+n)-p^2-pg-g^2}{18m(p^2+pg+g^2)}\sum_{cyc}[(p-g)ab-(2p+g)bc+(p+2g)ca]^2\ge 0$$

From now, we can easily check that if $\begin{cases} m>0 \\ 3m(m+n)\ge p^2+pg+g^2 \end{cases}$ then the inequality is true.

Our theorem is proved. J

II. Application.

Example 1. (Vasile Cirtoaje) Prove that

$$(a^2+b^2+c^2)^2\ge 3(a^3b+b^3c+c^3a).$$

Solution.

The inequality is equivalent to

$$\sum_{cyc}a^4+2\sum_{cyc}a^2b^2-\sum_{cyc}a^3b\ge 0$$

From this, we get $m=1, n=2, p=-3, g=0$, we have

$$\begin{cases} m=1>0 \\ 3m(m+n)-p^2-pg-g^2=3\cdot 1\cdot(1+2)-(-3)^2-(-3)\cdot 0-0^2=0 \end{cases}$$

Then using our theorem, the inequality is proved. J

Example 2. (Võ Quốc Bá Cẩn) Prove that

$$a^4+b^4+c^4+\left(\sqrt{3}-1\right)abc(a+b+c)\ge\sqrt{3}(a^3b+b^3c+c^3a).$$

Solution.

We have $m=1, n=0, p=-\sqrt{3}, g=0$ and

$$\begin{cases} m=1>0 \\ 3m(m+n)-p^2-pg-g^2=3\cdot 1\cdot(1+0)-\left(-\sqrt{3}\right)^2-\left(-\sqrt{3}\right)\cdot 0-0^2=0 \end{cases}$$

Then the inequality is proved. J

Example 3. (Phạm Văn Thuận) Prove that

$$7(a^4+b^4+c^4)+10(a^3b+b^3c+c^3a)\ge 0.$$

Solution.

We will prove the stronger result, that is

$$7\sum_{cyc}a^4+10\sum_{cyc}a^3b\ge\frac{17}{27}\left(\sum_{cyc}a\right)^4$$

$$\Leftrightarrow 86\sum_{cyc} a^4 - 51\sum_{cyc} a^2b^2 + 101\sum_{cyc} a^3b - 34\sum_{cyc} ab^3 - 102\sum_{cyc} a^2bc \ge 0$$

$$\Rightarrow \begin{cases} m = 86 \\ n = -51 \\ p = 101 \\ g = -34 \end{cases}$$

Moreover

$$\begin{cases} m = 86 > 0 \\ 3m(m+n) - p^2 - pg - g^2 = 3\cdot 86\cdot(86-51) - 101^2 - 101\cdot(-34) - (-34)^2 = 1107 > 0 \end{cases}$$

Then the inequality is proved. J

Example 4. (Vũ Đình Quý) Let $a,b,c > 0, abc = 1.$ Prove that

$$\frac{1}{a^2-a+1} + \frac{1}{b^2-b+1} + \frac{1}{c^2-c+1} \le 3.$$

Solution.

On Mathlinks inequality forum, I posted the following proof:

Lemma. If $a,b,c > 0, abc = 1,$ then $\frac{1}{a^2+a+1} + \frac{1}{b^2+b+1} + \frac{1}{c^2+c+1} \ge 1.$

Proof. From the given condition $a,b,c > 0, abc = 1$, there exist $x,y,z > 0$ such that $\begin{cases} a = \frac{yz}{x^2} \\ b = \frac{zx}{y^2} \\ c = \frac{xy}{z^2} \end{cases}$. And then, the inequality becomes

$$\sum_{cyc} \frac{x^4}{x^4 + x^2yz + y^2z^2} \ge 1$$

By the Cauchy Schwarz Inequality, we get

$$\sum_{cyc} \frac{x^4}{x^4 + x^2yz + y^2z^2} \ge \frac{\left(\sum_{cyc} x^2\right)^2}{\sum_{cyc}(x^4 + x^2yz + y^2z^2)} = \frac{\left(\sum_{cyc} x^2\right)^2}{\sum_{cyc} x^4 + \sum_{cyc} y^2z^2 + \sum_{cyc} x^2yz} \ge \frac{\left(\sum_{cyc} x^2\right)^2}{\sum_{cyc} x^4 + 2\sum_{cyc} y^2z^2} = 1$$

Our lemma is proved.

Now, using our lemma with note that $\begin{cases} \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2} > 0 \\ \frac{1}{a^2}\cdot\frac{1}{b^2}\cdot\frac{1}{c^2} = 1 \end{cases}$, we get

$$\sum_{cyc}\frac{x^4}{x^4+x^2+1}\ge 1 \Leftrightarrow \sum_{cyc}\frac{x^2+1}{x^4+x^2+1}\le 2 \Leftrightarrow \sum_{cyc}\frac{2(x^2+1)}{x^4+x^2+1}\le 4$$

$$\Leftrightarrow \sum_{cyc}\frac{(x^2+x+1)+(x^2-x+1)}{(x^2+x+1)(x^2-x+1)}\le 4 \Leftrightarrow \sum_{cyc}\frac{1}{x^2-x+1}+\sum_{cyc}\frac{1}{x^2+x+1}\le 4$$

Using our lemma again, we can get the result. J

Now, I will present another proof of mine based on this theorem

Since $a,b,c>0, abc=1,$ there exists $x,y,z>0$ such that $a=\frac{y}{x}, b=\frac{z}{y}, c=\frac{x}{z}$ then our inequality becomes

$$\sum_{cyc}\frac{x^2}{x^2-xy+y^2}\le 3 \Leftrightarrow \sum_{cyc}\frac{3x^2}{x^2-xy+y^2}\le 9 \Leftrightarrow \sum_{cyc}\left(4-\frac{3x^2}{x^2-xy+y^2}\right)\ge 3 \Leftrightarrow \sum_{cyc}\frac{(x-2y)^2}{x^2-xy+y^2}\ge 3$$

By the Cauchy Schwarz Inequality, we get

$$\left[\sum_{cyc}\frac{(x-2y)^2}{x^2-xy+y^2}\right]\left[\sum_{cyc}(x-2y)^2(x^2-xy+y^2)\right]\ge\left[\sum_{cyc}(x-2y)^2\right]^2$$

It suffices to show that

$$\left[\sum_{cyc}(x-2y)^2\right]^2\ge 3\sum_{cyc}(x-2y)^2(x^2-xy+y^2)$$

$$\Leftrightarrow 10\sum_{cyc}x^4+39\sum_{cyc}x^2y^2-25\sum_{cyc}x^3y-16\sum_{cyc}xy^3-8\sum_{cyc}x^2yz\ge 0$$

From this, we get $m=10, n=39, p=-25, g-16$ and

$$\begin{cases} m=10>0 \\ 3m(m+n)-p^2-pg-g^2=3\cdot 10\cdot(10+39)-(-25)^2-(-25)\cdot(-16)-(-16)^2=189>0 \end{cases}$$

Then using our theorem, the inequality is proved. J

III. Some problems for own study.

<u>Problem 1</u>. (Vasile Cirtoaje) Prove that

$$a^4+b^4+c^4+a^3b+b^3c+c^3a\ge 2(a^3b+b^3c+c^3a).$$

<u>Problem 2</u>. (Phạm Văn Thuận, Võ Quốc Bá Cẩn) Prove that

$$a(a+b)^3+b(b+c)^3+c(c+a)^3\ge\frac{8}{27}(a+b+c)^4.$$

<u>Problem 3</u>. (Phạm Kim Hùng) Prove that

$$a^4+b^4+c^4+\frac{1}{3}(ab+bc+ca)^2\ge 2(a^3b+b^3c+c^3a).$$

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Delicated to all members!