On the axiomatics of projective and affine geometry in terms of line intersection

Hans Havlicek* Victor Pambuccian

Abstract

By providing explicit definitions, we show that in both affine and projective geometry of dimension ≥ 3, considered as first-order theories axiomatized in terms of lines as the only variables, and the binary line-intersection predicate as primitive notion, non-intersection of two lines can be positively defined in terms of line-intersection.
Mathematics Subject Classification (2000): 51A05, 51A15, 03C40, 03B30.
Key Words: line-intersection, projective geometry, affine geometry, positive definability, Lyndon's preservation theorem.

M. Pieri [15] first noted that projective three-dimensional space can be axiomatized in terms of lines and line-intersections. A simplified axiom system was presented in [7], and two new ones in [17] and [10], by authors apparently unaware of [15] and [7]. Another axiom system was presented in [16, Ch. 7], a book devoted to the subject of three-dimensional projective line geometry.

One of the consequences of [4] is that axiomatizability in terms of line-intersections holds not only for n-dimension projective geometry with $n = 3$, but for all $n \geq 3$. Two such axiomatizations were carried out in [14]. It follows from [5] that there is *more* than just plain axiomatizability in terms of line-intersections that can be said about projective geometry, and it is the purpose of this note to explore the statements that can be made inside these theories, or in other words to find the definitional equivalent for the theorems of Brauner [2], Havlicek [5], and Havlicek [6], which state that *bijective* mappings between the line sets of projective or affine spaces of the same dimension ≥ 3 which map intersecting lines into intersecting lines stem from collineations, or, for three-dimensional projective spaces, from correlations. (See also [1, Ch. 5], [9], and [11]).

We shall also prove that, in the projective case, for $n \geq 4$, 'bijective' can be replaced by 'surjective' in the above theorem, and the same holds in the affine case for $n \geq 3$.

Let $\mathcal{L}$ denote the one-sorted first-order language, with individual variables to be interpreted as *lines*, containing as only non-logical symbol the binary relation symbol $\sim$, with $a \sim b$ to be interpreted as 'a intersects b' (and thus are *different* lines).

Given Lyndon's preservation theorem ([13], see also [8, Cor. 10.3.5, p. 500])—

Theorem. *Let $\mathcal{L}$ be a first order language containing a sign for an identically false formula, $\mathcal{T}$ be a theory in $\mathcal{L}$, and $\varphi(\mathbf{X})$ be an $\mathcal{L}$-formula in the free variables $\mathbf{X} = (X_1, \ldots, X_n)$. Then the following assertions are equivalent:*

*Corresponding author.

(i) there is a positive $\mathcal{L}$-formula $\psi(\mathbf{X})$ such that $\mathcal{T} \vdash \varphi(\mathbf{X}) \leftrightarrow \psi(\mathbf{X})$;

(ii) for any $\mathfrak{A}, \mathfrak{B} \in Mod(\mathcal{T})$, and each epimorphism $f : \mathfrak{A} \to \mathfrak{B}$, the following condition is satisfied: if $\mathbf{c} \in \mathfrak{A}^n$ and $\mathfrak{A} \models \varphi(\mathbf{c})$, then $\mathfrak{B} \models \varphi(f(\mathbf{c}))$.

—there must exist a positive $\mathcal{L}$-definition for the non-intersection of two lines (note that our 'sign for an identically false formula' is $a \sim a$).

1 Projective Spaces

1.1 Dimension ≥ 4

We start with projective geometry of dimension $n \geq 4$. We shall henceforth write $a \simeq b$ for $a \sim b \vee a = b$, as well as $(a_1, \ldots, a_p \sim b_1, \ldots, b_q)$ for $\bigwedge_{1 \leq i \leq p, 1 \leq j \leq q} a_i \sim b_j$.
We first define the ternary co-punctuality predicate S, with $S(abc)$ standing for 'a, b, c are three different lines passing through the same point' by (addition in the indices, whenever the stated bound for the index variable is exceeded, is mod 3 throughout the paper)

$$S(a_1a_2a_3) \;:\Leftrightarrow\; (\forall\, g)(\exists\, h)\, g \sim h \wedge \Big(\bigwedge_{i=1}^{3}(a_i \sim a_{i+1}, h)\Big). \tag{1}$$

It is easy to see that (1) holds when the lines a_i are different and concurrent. Should the three lines a_i intersect pairwise in three different points, then they would be coplanar and, by $n \geq 4$, for a line g which is skew to that plane, we could not find an appropriate line h. Next we define the closely related ternary predicate $\overline{S}$, where $\overline{S}(abc)$ stands for 'c passes through the intersection point of a and b' by

$$\overline{S}(abc) \;:\Leftrightarrow\; S(abc) \vee \Big(a \sim b \wedge (c = a \vee c = b)\Big), \tag{2}$$

and then the quaternary predicate $\#$, with $ab \# cd$ to be read as 'the intersection point of a and b is different from that of c and d' by

$$a_1b_1 \# a_2b_2 \;:\Leftrightarrow\; (\forall\, g)(\exists\, h_1h_2) \bigwedge_{i=1}^{2}(a_i \sim b_i) \wedge \Big(\Big(\bigwedge_{i=1}^{2} \overline{S}(a_ib_ih_i) \wedge S(h_1h_2g)\Big) \vee \Big(\bigvee_{i=1}^{2} \overline{S}(a_ib_ig)\Big)\Big). \tag{3}$$

In fact, suppose that $P_1 := a_1 \cap b_1$ and $P_2 := a_2 \cap b_2$ are points and that g is a line. If P_1 or P_2 is on g, then the existence of h_1 and h_2 is trivial. If P_1 and P_2 are not on g (figure 1), then for $P_1 \neq P_2$ there exists a point $H \in g$ which is not on $\langle P_1, P_2 \rangle$, i. e. the line joining P_1 and P_2; hence the lines $h_i := \langle P_i, H \rangle$ ($i = 1, 2$) have the required properties. On the other hand, if $P_1 = P_2 \notin g$, then (3) cannot be satisfied, since $S(h_1, h_2, g)$ would imply $h_1 \neq h_2$, but $\overline{S}(a_i, b_i, h_i)$ would force $h_1 = h_2$.
Notice that we can now define positively the negation of line equality by

$$a \neq b \;:\Leftrightarrow\; (\exists\, g)\, ag \# bg, \tag{4}$$

which proves that a surjective map between the sets of lines of two projective spaces of dimension $n \geq 4$, which maps intersecting lines into intersecting lines, must be injective as well.

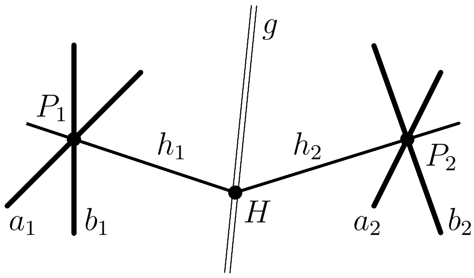

Figure 1.

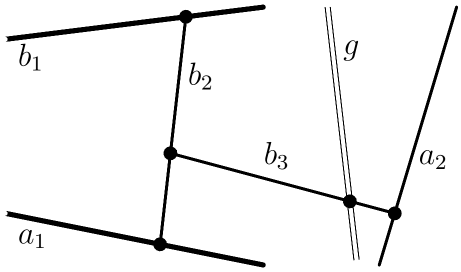

Figure 2.

We are now ready to define the non-intersection predicate $\not\sim$ for n-dimensional projective spaces with $n \geq 4$. Let $m = \left[\frac{n-1}{2}\right]$. For n even we have

$$a_1 \not\sim b_1 \quad :\Leftrightarrow \quad (a_1 = b_1) \vee (\exists\, a_2 \dots a_m)(\forall\, g)(\exists\, b_2 \dots b_{m+1}) \bigwedge_{i=2}^{m+1} b_i a_{i-1} \,\#\, b_i b_{i-1} \wedge g \sim b_{m+1}, \tag{5}$$

and for n odd we have

$$a_1 \not\sim b_1 \quad :\Leftrightarrow \quad (a_1 = b_1) \vee (\exists\, a_2 \dots a_m)(\forall\, g)(\exists\, b_2 \dots b_{m+1} c_2 \dots c_{m+1}) \bigwedge_{i=2}^{m+1} (b_i a_{i-1} \,\#\, b_i b_{i-1} \wedge c_i a_{i-1} \,\#\, c_i c_{i-1}) \wedge b_{m+1} g \,\#\, c_{m+1} g. \tag{6}$$

These two definitions state that, if a_1 does not intersect b_1, and if $a_1 \neq b_1$, then the set $\{a_1, b_1\}$ can be extended to a linearly independent set $A := \{b_1, a_1, \dots a_m\}$ (note than if $n = 4$, then $m = 1$, so there are no a's bound by the existential quantifier in (5) at all) spanning a subspace U of dimension $2m + 1$, i. e. the whole projective space if n is odd, or a hyperplane if n is even (see [3, II.5]). In both cases, any line g can be reached from A in the manner described in (5) and (6), as g lies in U if n is odd, and thus has two different points common with it, so (6) holds, and g intersects U in at least one point if n is even, so (5) holds. See figure 2 for the case $n = 6$.

If a_1 intersects a_2, then the dimension of the subspace U spanned by any A containing a_1 and a_2 will be, for n even, at most $n - 2$, so there are lines g which do not intersect U, and thus cannot be reached in the manner described in (5), and if n is odd, the dimension of U is at most $n - 1$, so there are lines g which intersects U in one point, so for those lines definition (6), which requires that the line g intersects U in two different points, cannot hold.

Given (1), (2), (3), it is obvious that n-dimensional projective geometry with $n \geq 3$, can be axiomatized inside $\mathcal{L}$, as one can rephrase the axiom system based on point line incidence of the Veblen-Young type (for example the one in Lenz [12, p. 19–20] to which lower- and upper-dimension axioms have been added) in terms of line intersections only, by replacing each 'point P' with two intersecting lines p_1 and p_2, the equality of two points P and Q, which have been replaced by (p_1, p_2) and (q_1, q_2), by $\overline{S}(p_1 p_2 q_1) \wedge \overline{S}(p_1 p_2 q_2)$ and every occurrence of 'P is incident with l' by $\overline{S}(p_1 p_2 l)$. This has been carried out in [14].

Since in some models (e. g. over commutative fields) of three-dimensional projective geometry there are correlations, S cannot be definable in terms of $\sim$, so the approach used for dimensions ≥ 4 fails in this case. However, $\not\sim$ is positively definable, with negated equality allowed, in terms of $\sim$, and it is to this definition that we now turn our attention.

1.2 The three-dimensional case

In the three-dimensional case, we first define the ternary relation T, with $T(abc)$ holding if and only if 'either the three different lines a, b, c intersect pairwise in three different points (and then we call abc a *tripod*) or they are concurrent, but do not lie in the same plane (in which case we call abc a *trilateral*)', by

$$T(a_1a_2a_3) \;:\Leftrightarrow\; (\forall\, g_1g_2)(\exists\, x_1x_2x_3)\,(g_1, g_2 \sim x_1, x_2, x_3) \\ \wedge \Big(\bigwedge_{i=1}^{3} \big((x_i \simeq a_i, a_{i+1}) \wedge a_i \sim a_{i+1}\big)\Big) \wedge \Big(\bigvee_{i=1}^{3} x_i \neq x_{i+1}\Big). \tag{7}$$

To see that the above definition holds when $a_1a_2a_3$ is a trilateral, let A_i be the point of intersection of the lines a_i and a_{i+1} for $i = 1, 2, 3$ (figure 3). Through each A_i there is a line x_i intersecting (and different from) both g_1 and g_2. The x_i satisfy the conditions of (7) since they cannot all coincide, given that no single line can, by the definition of a trilateral, pass through A_1, A_2, A_3. A dual reasoning to that presented for the case in which $a_1a_2a_3$ is a trilateral shows that the definition (7) holds for tripods $a_1a_2a_3$ as well.

To see that the only other case that could occur, given that $a_i \sim a_j$ for all $i \neq j$, namely that in which the three lines a_1, a_2, a_3 are lying in the same plane π and have a point O in common, does not satisfy (7), we choose g_1, g_2 such that they are skew, not in π, and intersect the line a_1 in two points that are different from O (figure 4). The only line that meets g_1, g_2 and two of the lines a_1, a_2, a_3 is a_1 itself.

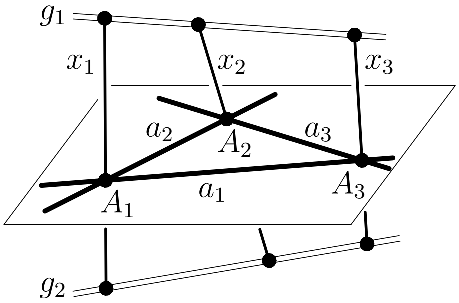

Figure 3.

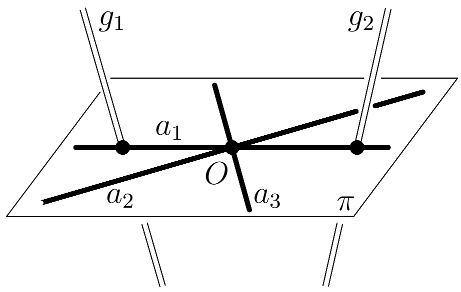

Figure 4.

Next, we define a sexternary predicate $\equiv_+$, with $abc \equiv_+ a'b'c'$ to be read as 'abc and $a'b'c'$ are either both trilaterals or both tripods', by

$$a_1b_1c_1 \equiv_+ a_2b_2c_2 \;:\Leftrightarrow\; (\forall\, g)(\exists\, x_{11}x_{21}x_{12}x_{22}x_{13}x_{23}) \\ \bigwedge_{i=1}^{2} \Big(T(a_ib_ic_i) \wedge \Big(\bigwedge_{j=1}^{3} (x_{ij} \simeq a_i, b_i, c_i, g) \wedge (x_{ij} \neq x_{i,j+1})\Big)\Big) \\ \wedge \Big(\bigwedge_{j=1}^{3} x_{1j} \simeq x_{2j}\Big). \tag{8}$$

Suppose that $a_1b_1c_1$ and $a_2b_2c_2$ are trilaterals in planes π_1 and π_2, respectively. Then the lines x_{ij} can be chosen as follows: If (i) $\pi_1 \neq \pi_2$ and if g is skew to the line $s = \pi_1 \cap \pi_2$, then we choose three distinct points X_1, X_2, X_3 on s, and we let x_{ij} be the line joining X_j with $g \cap \pi_i$ (figure 5). If (ii) $\pi_1 \neq \pi_2$ and if g and s are not skew, then we choose G to be a point lying on both g and s, and we let $x_{11} = x_{21} = s$, and choose for x_{i2} and x_{i3} any two distinct lines through G in the plane π_i, which are different from s. If (iii) $\pi_1 = \pi_2 = \pi$, then we let $x_{11} = x_{21}$, $x_{12} = x_{22}$, and $x_{13} = x_{23}$ be any three distinct lines in π through a point common to π and g. In case both $a_1b_1c_1$ and $a_2b_2c_2$ are tripods, the reasoning is, by dint of duality, similar.

Should $a_1b_1c_1$ be a trilateral in a plane π, and $a_2b_2c_2$ be a tripod with the vertex (point of concurrence) P, then we let g be a line which neither passes through P nor lies in π (figure 6). Let G be the point of intersection of g with π, and let γ be the plane spanned by g and P. If lines x_{ij} were to satisfy the conditions in the second line of (8), then $G \in x_{1j} \subset \pi$ and $P \in x_{2j} \subset \gamma$, and since at least two of the lines x_{1j}, say x_{11} and x_{12}, must be different from $\pi \cap \gamma$, the conditions $x_{11} \simeq x_{21}$ and $x_{12} \simeq x_{22}$ imply that both x_{21} and x_{22} have to be the line joining P with G, so they cannot be different, as required by the definiens in (8).

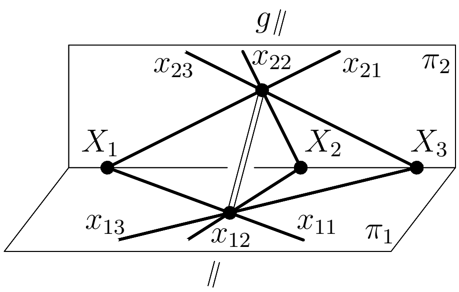

Figure 5.

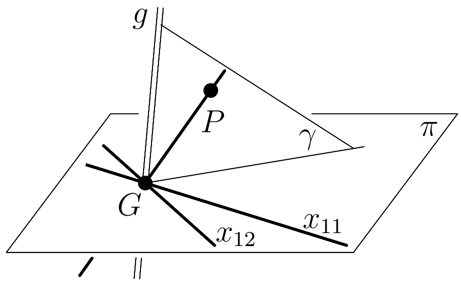

Figure 6.

We now define the sexternary predicate $\equiv_-$, with $abc \equiv_- a'b'c'$ standing for 'abc and $a'b'c'$ are (in any order) a trilateral and a tripod', by

$$a_1b_1c_1 \equiv_- a_2b_2c_2 \quad :\Leftrightarrow \quad (\forall\, g)(\exists\, x_1x_2) \bigwedge_{i=1}^{2} \Big((x_i \simeq a_i, b_i, c_i) \wedge T(a_ib_ic_i) \Big) \tag{9}$$
$$\wedge \Big(\bigvee_{i=1}^{2} (g = x_i \vee a_ib_ic_i \equiv_+ gx_1x_2) \Big).$$

Suppose $a_1b_1c_1$ is a trilateral, lying in the plane π, and $a_2b_2c_2$ is a tripod, with vertex P. If g is a line in π then we choose $x_1 = g$ and as x_2 any line through P. The case that g passes through P can be treated similarly. Hence we may restrict our attention to the case in which g neither lies in π nor passes through P, and denote in this case by G the point of intersection of g and π, and by γ the plane spanned by P and g.

Then (i) if $P \notin \pi$, we let x_2 be the line joining P and G, and x_1 be any line in π passing through G and different from the line $\pi \cap \gamma$ (figure 7), and (ii) if $P \in \pi$, then we let x_2 be the line joining P with G, and we let x_1 be any line in π passing through G, but different from x_2 (figure 8).

Now if both $a_1b_1c_1$ and $a_2b_2c_2$ were trilaterals lying in the same plane, then for any line g not lying in that plane, we could not find x_1 and x_2 with the desired properties, as the requirement that $\bigwedge_{i=1}^{2}(x_i \simeq a_i, b_i, c_i)$ forces them to lie in π, and so they can neither be equal to g nor form a trilateral with it. If both $a_1b_1c_1$ and $a_2b_2c_2$ were trilaterals lying in different planes π_1 and π_2, whose line of intersection is l, then for any line g intersecting l but lying neither in π_1 nor in π_2, we could not find the desired x_1 and x_2, as the condition $\bigwedge_{i=1}^{2}(x_i \simeq a_i, b_i, c_i)$ forces them to lie in π_1 and π_2, so they can neither be equal to g, nor from a trilateral with it. A dual reasoning shows that, if $a_1b_1c_1$ and $a_2b_2c_2$ were both tripods, (9) could not hold.

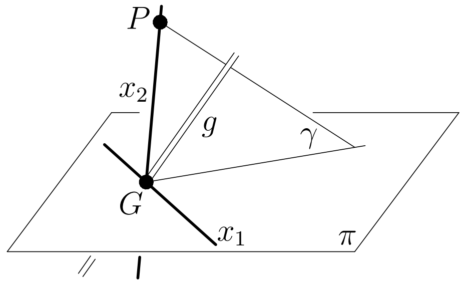

Figure 7.

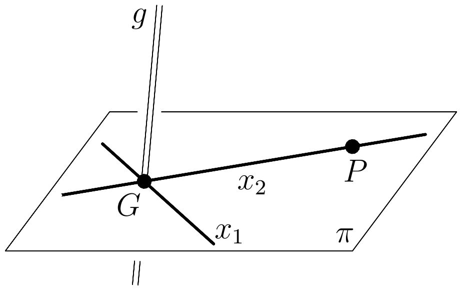

Figure 8.

The sexternary predicate $\equiv_\oplus$, with $abc \equiv_\oplus a'b'c'$ standing for 'abc and $a'b'c'$ are both trilaterals lying in *different* planes or both tripods with *different* vertices', is defined by

$$a_1b_1c_1 \equiv_\oplus a_2b_2c_2 \quad :\Leftrightarrow \quad (\exists\, x_1x_2x_3)\, a_1b_1c_1 \equiv_+ a_2b_2c_2 \wedge (x_3 \sim a_1, b_1, c_1, a_2, b_2, c_2) \qquad (10)$$
$$\wedge\, a_1b_1c_1 \equiv_- x_1x_2x_3 \wedge \Big(\bigwedge_{i=1}^{2}(x_i \sim a_i, b_i, c_i)\Big).$$

If $a_1b_1c_1$ and $a_2b_2c_2$ are both trilaterals (the tripod case is treated dually), lying in different planes π_1 and π_2 intersecting in g, then we choose a point P on g as the vertex of a tripod $x_1x_2x_3$, where $x_3 = g$, x_1 lies in π_1, and x_2 lies in π_2. If $a_1b_1c_1$ and $a_2b_2c_2$ were both trilaterals lying in the same plane π, then any x_1, x_2, x_3 satisfying the intersection conditions of (10) would have to belong to π, and thus could not form a tripod.

We are finally ready to define positively, with $\neq$ allowed, the skewness predicate σ, with $\sigma(ab)$ to be read 'the lines a and b are skew', by

$$\sigma(ab) \quad :\Leftrightarrow \quad (\forall\, g)(\exists\, xa_1a_2b_1b_2)\,(x \sim a, b) \wedge (x \simeq g) \qquad (11)$$
$$\bigwedge_{i=1}^{2}(aa_ix \equiv_+ bb_ix \wedge aa_ix \equiv_\oplus bb_ix) \wedge aa_1x \equiv_- aa_2x.$$

Suppose a and b are skew, and let P be a point on a (figure 9). The line g must have a point R in common with the plane determined by P and b. Let x be a line containing P, R and intersecting b in a point Q. Let a_1 be any line through P that does not lie in plane determined by a and x, a_2 be any line intersecting both x and a in points different from P, b_1 a line through Q not in the plane determined by b and x, and b_2 a line intersecting b and x in points different from Q. With these choices the definiens in (11) is satisfied.

Should a intersect b, and should g be chosen such that abg forms a tripod with vertex P, then, given that $(x \sim a, b) \wedge (x \simeq g)$, the x required to exist by (11) would have to pass through P. Since $aa_1x \equiv_- aa_2x$, one of aa_1x or aa_2x must be a tripod. W. l. o. g. we may suppose aa_1x is a tripod. By $aa_1x \equiv_+ bb_1x$, bb_1x must be a tripod as well, and by $aa_1x \equiv_\oplus bb_1x$ the two tripods must have different vertices, which is impossible, for, regardless of the choice of a_1 and b_1, the vertex of both tripods is P.

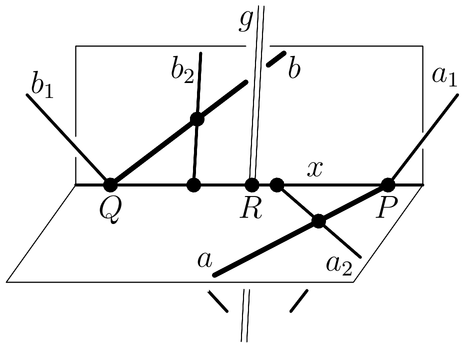

Figure 9.

The positive definition (in terms of $\sim$ with $\neq$ allowed) of the non-intersection predicate we were looking for in the three-dimensional case is

$$a \not\sim b \;:\Leftrightarrow\; a = b \vee \sigma(ab). \tag{12}$$

However, we do not know whether $\neq$, the negated line equality, is positively definable in terms of $\sim$, and thus whether it is possible to have a thoroughly positive definiens in (12).

2 Affine spaces

Notice that (1)–(4) are valid in n-dimensional affine geometry with $n \geq 3$ as well, since for any plane there is a disjoint parallel line.

Since (4) holds, any surjective map between the sets of lines of two affine spaces of dimension $n \geq 3$, which maps intersecting lines into intersecting lines must be injective as well.

In affine geometry, we distinguish two cases: (A) the one in which every line is incident with exactly two points (and then the space can be coordinatized by $\mathrm{GF}(2)$), and (B) the one in which every line is incident with at least three points. The number of all lines is $k := 2^{n-1}(2^n - 1)$ in case (A), whereas in case (B) this number is strictly greater than k. Hence we can characterize cases (A) and (B) by

$$\alpha \;:\Leftrightarrow\; (\forall\, x_1 \ldots x_{k+1})\, \Big(\bigvee_{1 \leq i < j \leq k+1} x_i = x_j \Big) \tag{13}$$

and $\neg\, \alpha$, respectively. It is worth noticing that the negated equalities in $\neg\alpha$ can be avoided altogether, without using (4), and that the number of variables in $\neg\alpha$ can be greatly reduced, by taking into account that in case (A) there are no more than $2^n - 1$ pairwise intersecting lines, namely all the lines through a fixed point, whereas in case (B) this number is exceeded. Therefore

$$\beta \;:\Leftrightarrow\; (\exists\, x_1 \ldots x_{2^n})\, \Big(\bigvee_{1 \leq i < j \leq 2^n} x_i \sim x_j \Big) \tag{14}$$

positively characterizes case (B).

Affine geometry can be axiomatized in terms of points and lines, with point-line incidence and line-parallelism as primitive notions, and the first such axiomatization was presented in [12, §2].

Affine geometry of a fixed dimension $n \geq 3$, in which (A) holds, cannot be axiomatized inside $\mathcal{L}$, as it is not possible to define the line-parallelism predicate $\parallel$ in terms of line-intersection, given that there are maps that preserve both $\sim$ and $\not\sim$, but which do not preserve $\parallel$, but it can be axiomatized in terms of lines, $\sim$, and $\parallel$. Affine geometry of a fixed dimension $n \geq 3$, in which (B) holds, can be axiomatized inside $\mathcal{L}$, by rephrasing the axiom system in [12, §2] in terms of lines and $\sim$ (this is possible in this case as $a \parallel b$ can be replaced by $\pi(ab) \wedge a \not\sim b$, where π is the coplanarity predicate defined below in (16)), and by adding suitable dimension axioms. However, regardless of whether (A) or its negation has been added to the axiom system of n-dimensional affine geometry with $n \geq 3$, it is true that $\not\sim$ can be defined positively in terms of $\sim$, given that $\neq$, which occurs in (15), can be defined positively by means of (4).

If every line contains exactly two points, i. e. in case (A), then it is quite easy to define positively the non-intersection predicate by observing that, if two different lines do not intersect, then there is more than one line that intersects the two lines in different points, but if they do intersect there is only one such line. Therefore the definition in this case is

$$a_1 \not\sim a_2 \;:\Leftrightarrow\; a_1 = a_2 \vee \left(\alpha \wedge (\exists\, b_1 b_2)\, b_1 \neq b_2 \wedge \Big(\bigwedge_{i=1}^{p} a_1 b_i \,\#\, a_2 b_i\Big)\right). \tag{15}$$

We denote the definiens of this definition by γ. The conjunct α in (15) is not needed if we regard it plainly as a definition of non-intersection inside the $\mathcal{L}$-theory of n-dimensional affine spaces over $\mathrm{GF}(2)$, but we shall use γ in the general case, where we have no information regarding the number of points incident with a line, below, and there we do need that conjunct as well.

From now on, we assume that lines are incident with more than two points. For all dimensions $n \geq 3$ we can define the coplanarity π of two lines (which are allowed to coincide) by

$$\pi(ab) \;:\Leftrightarrow\; (\exists\, cde) S(acd) \wedge S(bce) \wedge d \sim b \wedge d \sim e \wedge e \sim a. \tag{16}$$

See figure 10.

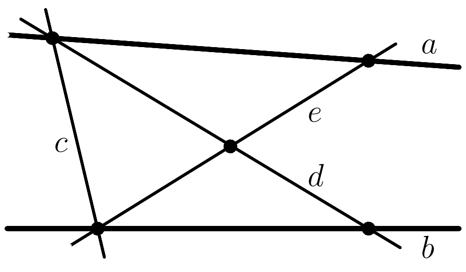

Figure 10.

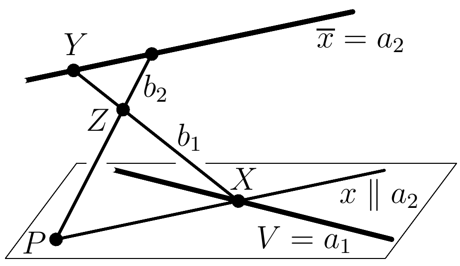

Figure 11.

To define non-intersection in n-dimensional affine space with $n \geq 3$, we need the following

Lemma. *Let $n \geq 3$, $m = [\frac{n+1}{2}]$, let $a_1, \ldots, a_m$ be m independent lines in n-dimensional affine space, let $U = \langle a_1, \ldots, a_m \rangle$ be the subspace spanned by these lines, and let $V = \langle a_1, \ldots, a_{m-1} \rangle$. Then for any point $P \in U$ there are (not necessarily distinct) lines b_1 and b_2, such that b_1 joins a point in V with a point on a_m, b_2 joins a point in V or in a_m with a different point on b_1, and P lies on b_2.*

Proof. If P is on a_m (or if $P \in V$), then choose $b_1 = b_2$ to be a line joining P with a point in V (or in a_m). If P is neither on a_m nor in V, then the subspaces $\langle P, a_m \rangle$ and $\langle P, V \rangle$ intersect in a line x. If x intersects both a_m and V in a point, then we let $b_1 = b_2 = x$. Since x cannot be parallel to both x and V, if it doesn't intersect both, it may be parallel to only one of them, i. e. either (i) $x \parallel V$ or (ii) $x \parallel a_m$. Let X be the point of intersection of x with (i) a_m or (ii) V. Let Y be a point in (i) V or (ii) a_m, let $\overline{x}$ be the parallel through Y to x, and $b_1 := \langle X, Y \rangle$. (Figure 11 depicts case (ii) for $m = 2$, so that $V = a_1$ and $\overline{x} = a_2$.) Let Z be a third point on b_1 and let $b_2 := \langle P, Z \rangle$. The line b_2 is not parallel to $\overline{x}$ and thus intersects (i) V or (ii) a_m in a point which is different from Z. $\square$

We now define some auxiliary predicates. Let $M(a_1 \dots a_m x)$ stand for 'x is one of the lines a_i or it intersects two of these lines in different points', i. e.

$$M(a_1 \dots a_m x) \;:\Leftrightarrow\; \Big(\bigvee_{i=1}^{m} x = a_i\Big) \vee \Big(\bigvee_{1 \le i < j \le m} a_i x \# a_j x\Big). \tag{17}$$

Closely related to M, we introduce

$$M_q(a_1 \dots a_m x) \;:\Leftrightarrow\; (\exists\, b_1 \dots b_q) \bigwedge_{i=1}^{q} M(a_1 \dots a_m b_1 \dots b_i) \wedge M(a_1 \dots a_m b_1 \dots b_q x). \tag{18}$$

If (18) holds then the line x belongs to the affine subspace spanned by $a_1, ..., a_m$, since it can be 'reached' with the help of the auxiliary lines $b_1, \dots, b_q$.

With m standing for $[\frac{n+1}{2}]$, whenever $a_1 \not\sim a_2$, we can find lines $a_3, \dots, a_m$ such that $a_1, \dots, a_m$ are independent. Let U be the subspace spanned by them. We infer from the above lemma, that each line h in U satisfies $M_r(a_1 \dots a_m h)$ for $r = 2^{m+1} - 4$. Recall that β ensures that we are in case (B). So we can now state the definition of non-intersection, when n is even (in this case U is a hyperplane, so that to any line g there exists a line h in U coplanar with g) as

$$a_1 \not\sim a_2 :\Leftrightarrow\; a_1 = a_2 \vee \Big(\beta \wedge (\exists\, a_3 \dots a_m)(\forall\, g)(\exists\, h)\, \pi(gh) \wedge M_r(a_1 \dots a_m h)\Big). \tag{19}$$

If n is odd, U is the whole affine space, so any line g lies in U, and thus

$$a_1 \not\sim a_2 :\Leftrightarrow\; a_1 = a_2 \vee \Big(\beta \wedge (\exists\, a_3 \dots a_m)(\forall\, g)\, M_r(a_1 \dots a_m g)\Big). \tag{20}$$

The definiens of the definitions in (19) and (20) are denoted by δ_0 and δ_1, respectively.

Finally, we return to the general case of n-dimensional affine geometry. By (15), (19), and (20) the definition of non-intersection is

$$a_1 \not\sim a_2 \;:\Leftrightarrow\; \gamma \vee \delta_{2\left(\frac{n}{2} - [\frac{n}{2}]\right)}. \tag{21}$$

3 Higher-dimensional subspaces

Given [4], n-dimensional projective geometry can also be axiomatized with k-dimensional subspaces (for all $1 \le k \le n-1$ with $2k+1 \ne n$) as individual variables and a binary intersection predicate $\sim$, with $a \sim b$ to be interpreted as 'the subspaces a and b intersect in a $k-1$-dimensional subspace'. From the results in [9] it follows that the non-intersection predicate is also positively definable in terms of the intersection predicate (negated equality is allowed), but the actual definition will very likely be prohibitively intricate.

References

[1] W. Benz, *Geometrische Transformationen*, BI Wissenschaftsverlag, Mannheim, 1992.

[2] H. Brauner, Über die von Kollineationen projektiver Räume induzierten Geradenabbildungen, *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **197** (1988), 327–332.

[3] W. Burau, *Mehrdimensionale projektive und höhere Geometrie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1961.

[4] W.-L. Chow, On the geometry of algebraic homogeneous spaces, *Ann. Math.* **50** (1949), 32–67.

[5] H. Havlicek, On isomorphisms of Grassmann spaces, *Mitt. Math. Ges. Hamburg* **14** (1995), 117–120.

[6] H. Havlicek, Isomorphisms of affine Plücker spaces, in: N. K. Artémiadis, N. K. Stephanidis, editors, *Proceedings of the 4th International Congress of Geometry (Thessaloniki, 1996)*, 171–178, Giachoudis-Giapoulis, Thessaloniki, 1997.

[7] E. R. Hedrick, L. Ingold, A set of axioms for line geometry, *Trans. Amer. Math. Soc.* **15** (1914), 205–214.

[8] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications 42, Cambridge University Press, Cambridge, 1993.

[9] W.-l. Huang, Adjacency preserving transformations of Grassmann spaces, *Abh. Math. Sem. Univ. Hamburg* **68** (1998), 65–77.

[10] E. Kożniewski, An axiom system for the line geometry, *Demonstratio Math.* **21** (1988), 1071–1087.

[11] A. Kreuzer, On isomorphisms of Grassmann spaces, *Aequationes Math.* **56** (1998), 243–250.

[12] H. Lenz, Zur Begründung der analytischen Geometrie, *S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss.* 1954, 17–72.

[13] R. C. Lyndon, Properties preserved under homomorphism, *Pacific J. Math.* **9** (1959), 143–154.

[14] V. Pambuccian, Elementary axiomatizations of projective space and its associated Grassmann space, submitted.

[15] M. Pieri, Sui principi che regono la geometria delle rette, *Atti Accad. Torino* **36** (1901), 335–350 (also in: *Opere scelte, Opere sui fondamenti della matematica*, pp. 273–288, UMI, Bologna 1980).

[16] H.-J. Stoß, *Treffgeraden und Nullinvarianz. Beiträge zur Liniengeometrie*, Philosophisch-Anthroposophischer Verlag am Goetheanum, Dornach, 1995.

[17] S. Trott, An axiomatic line geometry, *Canad. J. Math.* **20** (1968), 939–952.

Hans Havlicek, Institut für Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8–10/1133, A–1040 Wien, Austria.

Victor Pambuccian, Department of Integrative Studies, Arizona State University West, P. O. Box 37100, Phoenix, AZ 85069-7100, U.S.A.