# The Interesting Around Technical Analysis Three Variable Inequalities

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As we knew, The three variable inequalities (TVI) is a section very interesting and hard in inequality. About 10 years back to now, the three variable inequality being 'fertile ground' in the inequality now. Because that it is very nice, onle three variable simple $a, b, c$ but we have very much inequality, very much interesting in its. Beside very hard problem is very method was born to solved TVI. But only method has interesting in its, Also we inevitable too weakness of only method. In this sections we 'll explore a method (Not new) that we thinks it very interesting and definitely it is very useful for you in contests Olympic Mathematical,.... And now we 'll enjoy it.

Learning Inequality, definitely who know Sum Of Square (SOS) and Vornicu-Schur (VS) method too, so we don't talk much for its. As we knew , with three variable are symmetry and cyclic permutation inequality we may performances them to SOS form $S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq 0$ or VS is $x(a-b)(a-c)+y(b-a)(b-c)+z(c-a)(c-b) \geq 0$.Firstly we will review them

**I. From SOS** : $S = f(a, b, c) = S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2$.
Including $S_a, S_b, S_c$ is functions have $a, b, c$ is variables.

**1**.If $S_a, S_b, S_c \geq 0$ then $S \geq 0$.
**2**.If $a \geq b \geq c$ and $S_b, S_b + S_c, S_b + S_a \geq 0$ then $S \geq 0$.
**3**. If $a \geq b \geq c$ and $S_b, S_c \geq 0, a^2S_b + b^2S_a \geq 0$ then $S \geq 0$.
**4**. If $a \geq b \geq c$ and $(a-c)S_b + (a-b)S_c \geq 0, (a-c)S_b + (b-c)S_a \geq 0$ then $S \geq 0$.
**5**.If $a \geq b \geq c$ and $S_b, S_c \geq 0, (a-c)S_b + (b-c)S_a \geq 0$ then $S \geq 0$.
**6**. If $S_a + S_b + S_c \geq 0$ and $S_aS_b + S_bS_c + S_cS_a \geq 0$ then $S \geq 0$.

We will prove them.

**1**. Of course that $(a-b)^2, (b-c)^2, (c-a)^2 \geq 0$ but $S_a, S_b, S_c \geq 0$ so $S \geq 0$.
**2**.Because $a \geq b \geq c$ so $(a-c)^2 = (a-b)^2 + (b-c)^2 + 2(a-b)(b-c) \geq (a-b)^2 + (b-c)^2$
So $S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 = (S_a + S_b)(b-c)^2 + (S_b + S_c)(a-b)^2$.
Because $S_b, S_b + S_c, S_b + S_a \geq 0$ so $S \geq 0$.
**3**. We have $(a-c) - \dfrac{a}{b}(b-c) = \dfrac{c(a-b)}{b} \geq 0$ and $S_c(a-b)^2 \geq 0$. So

$$S = S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \geq S_a(b-c)^2+S_b.\frac{a^2}{b^2}(b-c)^2 = \frac{a^2S_b + b^2S_a}{b}(b-c)^2 \geq 0.$$

**4,5**. We have $S=(S_a+S_b)(b-c)^2+(S_b+S_c)(a-b)^2$

$$=(b-c)[(S_a+S_b)(b-c)+S_b(a-b)]+(a-b)[(S_b+S_c)(a-b)+S_b(b-c)].$$

$$=(b-c)[(a-c)S_b+(b-c)S_a]+(a-b)[(a-c)S_b+(a-b)S_c]\geq 0.$$

**6**.Notive that if $S_a+S_b+S_c\geq 0$ then we can assume that $S_a+S_b\geq 0$. Let $x=a-b, y=b-c$ then

$$S_cx^2+S_ay^2+S_b(x+y)^2=(S_a+S_b)y^2+2S_bxy+(S_b+S_c)x^2$$

Because

$$\Delta'=S_b^2-(S_b+S_a)(S_b+S_c)=-(S_aS_b+S_bS_c+S_cS_a)\leq 0$$

So we are done.

**II. Form VS:** For $a\geq b\geq c$ and $x,y,z$ be non-negative function , See that inequality

$$V=x\left(a-b\right)\left(a-c\right)+y\left(b-c\right)\left(b-a\right)+z\left(c-a\right)\left(c-b\right)\geq 0$$

This inequality is true if it such that one condition in that conditions

**1**. $x\geq y$ .
**2**. $z\geq y$.
**3**. $x+z\geq y$
**4**. $\sqrt{x}+\sqrt{z}\geq\sqrt{y}$.
**5**.$ax\geq by$ (If $a,b,c$ be nonnegative real numbers and $a\geq b$.)
**6**. $cz\geq by$ (If $a,b,c$ be nonnegative real numbers and $a\geq b$.)
**7**. $ax+cz\geq by$ (If $a,b,c$ be nonnegative real numbers and $a\geq b$.)
**8**. $\sqrt{ax}+\sqrt{cz}\geq\sqrt{by}$ with $a\geq b\geq c\geq 0$.
**9**. $bz\geq cy$ with $a,b,c$ be sides of a triangle.
Now, we 'll prove them.

See that in 9 case then **1,2,3** was rightarrow from **4** and **5,6,7** was rightarrow from **8** so we need prove **4,8,9**.Indeed ,

**4**.We have

$$V=x\left(a-b\right)\left(a-c\right)+y\left(b-c\right)\left(b-a\right)+z\left(c-a\right)\left(c-b\right)$$

$$=\left[\sqrt{x}a-\left(\sqrt{x}+\sqrt{z}\right)b+\sqrt{z}c\right]^2+\left[\left(\sqrt{x}+\sqrt{z}-y\right)\right](a-b)(b-c).$$

So $V\geq 0$.

**8**. Case $a=b$ or $b=c$ then the inequality is of course. Let case $a>b>c\geq 0$.
Multiply two hands with $(a-b)(b-c)>0$ we have the inequality is equivalent to

$$x\frac{a-c}{b-c}+z\frac{a-c}{a-b}\geq y.$$

Apply AM-GM Inequality we have

$$LHS=\frac{ax+cz}{b}+x\left(\frac{a-c}{b-c}-\frac{a}{b}\right)+z\left(\frac{a-c}{a-b}-\frac{c}{b}\right)=\frac{ax+cz}{b}+\frac{xc(a-b)}{b(b-c)}+\frac{za(b-c)}{b(a-b)}$$

$$=\frac{ax+cz}{b}+2\sqrt{\frac{xc(a-b)}{b(b-c)}.\frac{za(b-c)}{b(a-b)}}=\frac{\left(\sqrt{ax}+\sqrt{cz}\right)^2}{b}.$$

Because $\sqrt{ax}+\sqrt{cz}\geq\sqrt{by}$ so this inequality is true.
**9**. We have $a-c-\dfrac{b}{c}(a-b)=\dfrac{(b-c)(b+c-a)}{c}\geq 0$. So

$$V\geq y(b-c)(b-a)+z(a-c)(b-c)\geq y(b-c)(b-a)+z.\frac{b}{c}(a-b)(b-c)=\frac{bz-cy}{c}(a-b)(b-c)\geq 0.$$

Also have many diferent standards you can see in solving problem.
And an interesting here is $SOS \Leftrightarrow VS(S\Leftrightarrow V)$. Indeed,

$$S=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2=S_a(b^2+c^2-2bc)+S_b(c^2+a^2-2ca)+S_c(a^2+b^2-2ab)$$

$$=\sum\left(S_b+S_c\right)a^2-\sum\left(2S_a.bc\right)$$

$$=\sum\left[\left(S_b+S_c\right)a^2+\left(S_b+S_c\right)bc-\left(S_b+S_c\right)ab-\left(S_b+S_c\right)ac\right]\geq 0.$$

$$=\left(S_b+S_c\right)\left(a-b\right)\left(a-c\right)+\left(S_c+S_a\right)\left(b-c\right)\left(b-a\right)+\left(S_a+S_b\right)\left(c-a\right)\left(c-b\right)$$

So we have $x=S_b+S_c, y=S_a+S_c, z=S_a+S_b$.
Thus we can use 9 standards for SOS or 6 standards for VS.
Some equality useful that

1. $a^2+b^2-2ab=(a-b)^2$.
2. $\dfrac{a}{b}+\dfrac{b}{a}-2=\dfrac{(a-b)^2}{ab}$.
3. $a^2+b^2+c^2-ab-bc-ca=\dfrac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=\sum(a-b)(a-c)$.
4. $a^3+b^3+c^3-3abc=\dfrac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=\sum a\sum(a-b)(a-c)$.
5. $(a+b)(b+c)(c+a)-8abc=a(b-c)^2+b(c-a)^2+c(a-b)^2=\sum[(a+b)(a-b)(a-c)]$.
6. $\sqrt{2(a^2+b^2)}-(a+b)=\dfrac{(a-b)^2}{a+b+\sqrt{2(a^2+b^2)}}$.
7. $\sum\dfrac{a}{b+c}-\dfrac{3}{2}=\sum\dfrac{(a-b)^2}{2(a+c)(b+c)}=\sum\left[\dfrac{a+b+2c}{(2(a+b)(b+c)(c+a)}\right](a-b)(a-c)$.

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As in practice, we see that have a big duration TVI can solved by two this assessment. The work solved problems we can see very much in different inequality books so in this section we 'll prove some selective inequality nice and intersting.

***Problem 1 .*** Let a, b, c, x, y, z be six real (not necessarily nonnegative) numbers. Assume that $a\geq b\geq c$. Also, assume that either $x\geq y\geq z$ or $x\leq y\leq z$. Then,

$$\left(\sum_{\text{cyc}}\left(a-b\right)\left(a-c\right)\right)\cdot\left(\sum_{\text{cyc}}x^2\left(a-b\right)\left(a-c\right)\right)\geq\left(\sum_{\text{cyc}}x\left(a-b\right)\left(a-c\right)\right)^2$$

***Solution:***
The inequality we have to prove rewrites as

$$\left(\sum_{\text{cyc}}\left(a-b\right)\left(a-c\right)\right)\cdot\left(\sum_{\text{cyc}}x^2\left(a-b\right)\left(a-c\right)\right)-\left(\sum_{\text{cyc}}x\left(a-b\right)\left(a-c\right)\right)^2\geq 0$$

But a straightforward calculation reveals that

$$\left(\sum_{\text{cyc}}(a-b)(a-c)\right)\cdot\left(\sum_{\text{cyc}}x^2(a-b)(a-c)\right)-\left(\sum_{\text{cyc}}x(a-b)(a-c)\right)^2$$
$$=-(a-b)(b-c)(a-c)\left((b-c)(y-z)^2+(c-a)(z-x)^2+(a-b)(x-y)^2\right)$$

Since $a-b\geq 0$, $b-c\geq 0$ and $a-c\geq 0$ (this is all because $a\geq b\geq c$), instead of proving that this product is $\geq 0$, it will be enough to show that

$$(b-c)(y-z)^2+(c-a)(z-x)^2+(a-b)(x-y)^2\leq 0$$

This is equivalent to

$$(b-c)(y-z)^2+(a-b)(x-y)^2\leq -(c-a)(z-x)^2$$

what rewrites as

$$(b-c)(y-z)^2+(a-b)(x-y)^2\leq (b-c)(z-x)^2+(a-b)(z-x)^2$$

However, since $b-c\geq 0$ and $a-b\geq 0$, this will become trivial once we succeed to show that $(y-z)^2\leq (z-x)^2$ and $(x-y)^2\leq (z-x)^2$. But this is equivalent to $|y-z|\leq |z-x|$ and $|x-y|\leq |z-x|$, what is actually true because we have either $x\geq y\geq z$ or $x\leq y\leq z$. This completes the proof of problem 1.

***General:*** Let $p$ and $q$ be real numbers such that $pq>0$, and let $a,b,c$ be non-negative real numbers. Prove that $S_0.S_{p+q}\geq S_p.S_q$ , where $S_k=\sum_{cyc}a^k(a-b)(a-c)$
***Problem 2(Darij Grinberg)***. If $p$ is an even nonnegative integer, then the inequality $\sum_{\text{cyc}}a^p(a-b)(a-c)\geq 0$ holds for arbitary reals $a,b,c$.

***Solution:*** Since the inequality in question is symmetric, we can WLOG assume that $a\geq b\geq c$. Since p is an even nonnegative integer, we have $p=2n$ for some nonnegative integer $n$.

Define a function sign by sign $t=\begin{cases}-1, & \text{if } t<0\\ 0, & \text{if } t=0\\ 1, & \text{if } t>0\end{cases}$
Define $x=\text{sign } a\cdot|a^n|$, $y=\text{sign } b\cdot|b^n|$, $z=\text{sign } c\cdot|c^n|$.
Then,

$$x^2=(\text{sign } a\cdot|a^n|)^2=(a^n)^2=a^{2n}=a^p$$

and similarly $y^2=b^p$ and $z^2=c^p$.
But $a\geq b\geq c$ yields $x\geq y\geq z$ (in fact, the function

$$f(t)=\text{sign } t\cdot|t^n|$$

is monotonically increasing on the whole real axis). Thus, we can apply problem 1 to get

$$\left(\sum_{\text{cyc}}(a-b)(a-c)\right)\cdot\left(\sum_{\text{cyc}}a^p(a-b)(a-c)\right)=\left(\sum_{\text{cyc}}(a-b)(a-c)\right)\cdot\left(\sum_{\text{cyc}}x^2(a-b)(a-c)\right)$$

$$\ge \left(\sum_{\text{cyc}} x\left(a-b\right)\left(a-c\right)\right)^2 \ge 0$$

By the same argument as in the proof for standards 1 of VS. We see that we can divide by $\sum_{\text{cyc}}\left(a-b\right)\left(a-c\right)$, and obtain $\sum_{\text{cyc}} a^p\left(a-b\right)\left(a-c\right)\ge 0$.

***Problem 1(Nguyen Duy Tung, Nguyen Huy Tung).*** If $a,b,c$ are nonnegative real numbers, then

$$a^4+b^4+c^4+abc(a+b+c)\ge \sum bc(b+c)\sqrt{b^2-bc+c^2}.$$

***Firstly Solution:***
First,we have:$(b+c)\sqrt{b^2-bc+c^2}\ge b^2+c^2 \Leftrightarrow bc(b-c)^2\ge 0$ and then,

$$(b+c)\sqrt{b^2-bc+c^2}-(b^2+c^2)=\frac{bc(b-c)^2}{(b+c)\sqrt{b^2-bc+c^2}+(b^2+c^2)}\le \frac{bc(b-c)^2}{2(b^2+c^2)}.$$

So we just need to prove a Even Stronger one:

$$\sum a^4+abc\sum a-\sum bc(b^2+c^2)\ge \sum \frac{b^2c^2(b-c)^2}{2(b^2+c^2)}$$

$$\Leftrightarrow \frac{1}{2}\sum(a^2+b^2-c^2)(a-b)^2\ge \sum \frac{b^2c^2(b-c)^2}{2(b^2+c^2)}.$$

$$\Leftrightarrow \frac{1}{2}\sum\left(a^2+b^2-c^2-\frac{a^2b^2}{a^2+b^2}\right)(a-b)^2\ge 0.$$

WLOG $a\ge b\ge c$ ,and then we have:

$$S_c=a^2+b^2-c^2-\frac{a^2b^2}{a^2+b^2}=\frac{a^4+a^2b^2+b^4}{a^2+b^2}-c^2\ge a^2-c^2\ge 0$$

$$S_b=a^2+c^2-b^2-\frac{a^2c^2}{a^2+c^2}=\frac{a^4+a^2c^2+c^4}{a^2+c^2}-b^2\ge a^2-b^2\ge 0$$

So, $S_c(a-b)^2\ge 0$

$$S_b(a-c)^2+S_a(b-c)^2\ge (S_a+S_b)(b-c)^2$$

$$=\left[\left(b^2+c^2-a^2-\frac{b^2c^2}{b^2+c^2}\right)+\left(a^2+c^2-b^2-\frac{a^2c^2}{a^2+c^2}\right)\right](b-c)^2$$

$$=c^4(b-c)^2\left(\frac{1}{a^2+c^2}+\frac{1}{b^2+c^2}\right)\ge 0.$$

Plus the two inequalities together,we have $\sum S_c(a-b)^2\ge 0$ and the proof is completed.

***Second Solution:***
WLOG, we may assume that $c=\min(a,b,c)$. Apply AM-GM Inequality we have

$$a\sqrt{a^2-ac+c^2}+b\sqrt{b^2-bc+c^2}\le \frac{a^2+a^2-ac+c^2}{2}+\frac{b^2+b^2-bc+c^2}{2}.$$

Therefore,

$$ac(a+c)\sqrt{a^2-ac+c^2}+bc(b+c)\sqrt{b^2-bc+c^2}\le \frac{(ac+c^2)(2a^2-ac+c^2)}{2}+\frac{(cb+c^2)(2b^2-bc+c^2)}{2}.$$

$$= \frac{2a^3c + 2b^3c + 2c^4 + c^2(a^2+b^2)}{2}$$

It suffices to prove that

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a+b)\sqrt{a^2-ab+b^2} + a^3c + b^3c + c^4 + c^2(a^2+b^2)/2.$$

$$\Leftrightarrow a^4 + b^4 - ab(a+b)\sqrt{a^2-ab+b^2} \geq (a^3c + b^3c - a^2bc - ab^2c) + \frac{c^2(a^2+b^2)}{2} - abc^2.$$

We have

$$a^4+b^4-ab(a+b)\sqrt{a^2-ab+b^2} = [a^4+b^4-ab(a^2+b^2)] - \left[ab(a+b)\sqrt{a^2-ab+b^2} - ab(a^2+b^2)\right].$$

$$= (a-b)^2(a^2+ab+b^2) - \frac{a^2b^2(a-b)^2}{(a+b)\sqrt{b^2-ab+b^2}+(a^2+b^2)} \geq (a-b)^2(a^2+ab+b^2) - \frac{a^2b^2(a-b)^2}{2(a^2+b^2)}.$$

And because $c = \min(a,b,c)$ so

$$(a^3c + b^3c - a^2bc - ab^2c) + \frac{c^2(a^2+b^2)}{2} - abc^2 = (a-b)^2 . \frac{c^2+2ca+2cb}{2} \leq \frac{5ab(a-b)^2}{2}.$$

So we need prove that

$$a^2 + ab + b^2 - \frac{a^2b^2}{2(a^2+b^2)} \geq \frac{5ab}{2} \Leftrightarrow 2(a^4+b^3) + 3a^2b^2 \geq 3a^3b + 3ab^3.$$

By AM-GM Inequality

$$a^4 + a^2b^2 \geq 2a^3b, \quad b^4 + b^2a^2 \geq 2ab^3.$$

And $a^4 + b^4 - (a^3b + ab^3) = (a-b)^2(a^2+ab+b^2) \geq 0$.
Thus add them we have $2(a^4+b^4) + 3a^2b^2 \geq 2(a^4+b^4) + 2a^2b^2 \geq 3(a^3b + ab^3)$.
We have done. The equality holds when $a = b = c$.

***Problem 4*** a) (Cezar Lupu). Let $a, b, c$ be positive real numbers. Prove that

$$\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} \leq \frac{1}{4} + \frac{4abc}{(a+b)(b+c)(c+a)}.$$

***Solution:***
WLOG, we may assume that $a \geq b \geq c > 0$. We rewrite the original inequality into the following form

$$\sum_{cyc} \left(\frac{a-b}{a+b}\right)^2 \geq \frac{2\sum_{cyc} c(a-b)^2}{(a+b)(b+c)(c+a)}.$$

or equivalently,

$$\sum_{cyc} \left[\frac{1}{(a+b)^2} - \frac{2c}{(a+b)(b+c)(c+a)}\right](a-b)^2 \geq 0.$$

Now, let $S_a = \dfrac{1}{(b+c)^2} - \dfrac{2a}{(a+b)(b+c)(c+a)}, S_b = \dfrac{1}{(c+a)^2} - \dfrac{2b}{(a+b)(b+c)(c+a)}$.

$$S_c = \frac{1}{(a+b)^2} - \frac{2c}{(a+b)(b+c)(c+a)}.$$

It follows from the given expression that

$$S_c = \frac{1}{(a+b)^2} - \frac{2c}{(a+b)(b+c)(c+a)} = \frac{(c-a)(c-b)}{(a+b)^2(b+c)(a+c)}.$$

Likewise, $S_a$ can be expressed as $S_a = \dfrac{(a-b)(a-c)}{(b+c)^2(a+c)(a+b)}$.
Nonetheless, due to the initial assumption, it is easy to see that $S_a, S_c \geq 0$.
The given inequality can be rephrased as

$$f(a,b,c) = S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0.$$

Using the fact that

$$S_b + S_a = \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} - \frac{2(a+b)}{(a+b)(b+c)(c+a)} = \left(\frac{1}{a+c} - \frac{1}{b+c}\right)^2.$$

and

$$S_b + S_c = \frac{1}{(a+c)^2} + \frac{1}{(a+b)^2} - \frac{2(b+c)}{(a+b)(b+c)(c+a)} = \left(\frac{1}{a+b} - \frac{1}{a+c}\right)^2.$$

Therefore, $S_a + S_b \geq 0, S_c + S_b \geq 0, S_b \leq 0.(a \geq b \geq c)$.
From this point, we might view the original inequality with watchful eyes

$$f(a,b,c) = (S_a + S_b)(b-c)^2 + (S_c + S_b)(a-b)^2 + 2S_b(a-b)(b-c)$$

$$= \frac{(a-b)^2(b-c)^2}{(a+b)^2(a+c)^2} + \frac{(b-c)^2(a-b)^2}{(b+c)^2(a+c)^2} - \frac{2(a-b)^2(b-c)^2}{(a+c)^2(a+b)(b+c)} = \frac{(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

Clearly, $f(a,b,c)$ is non-negative so we complete our proof here.
***4b)*** Let $a, b, c$ are three positive reals ,prove that ,

$$\frac{b+c-a}{5a^2+4bc} + \frac{c+a-b}{5b^2+4ca} + \frac{a+b-c}{5c^2+4ab} \geq \frac{1}{a+b+c}$$

Solution,

$$\sum \frac{b+c-a}{5a^2+4bc} - \frac{1}{a+b+c} = (b-c)^2\left(\frac{5(a+b)-4c}{(5a^2+4bc)(5b^2+4ac)}\right) + \frac{c}{5a^2+4bc} + \frac{c}{5b^2+4ac} - \frac{6c^2-(a-b)^2}{(a+b+c)(5c^2+4ab)}$$

$$\geq \frac{c(6(a-b)^2+5a^2+5b^2+4(a+b)c)}{(5a^2+4bc)(5b^2+4ac)} - \frac{6c^2}{(a+b+c)(5c^2+4ab)}$$

It suffice to prove that

$$\frac{6(a-b)^2+5a^2+5b^2+4(a+b)c}{(5a^2+4bc)(5b^2+4ac)} \geq \frac{6c}{(a+b+c)(5c^2+4ab)}$$

Use the inequality

$$(5a^2+4bc)(5b^2+4ac) \geq \left(\frac{5a^2+5b^2+4ac+4bc}{2}\right)^2$$

And $(c-a)(c-b) \geq 0$ ,or $c^2+ab \geq ca+cb$

$$(a+b+c)(4a+4b+c)[5(a^2+b^2)+4(a+b)c+5(a-b)^2] - 6\left(\frac{5(a^2+b^2)+4(a+b)c}{2}\right)^2$$

$$= \left[3(5\left(\frac{a+b}{2}\right)^2 + 2(a+b)c\right] + \left(\frac{a+b-2c}{2}\right)^2 \left[10\left(\frac{a+b}{2}\right)^2 + 4(a+b)c + 34\left(\frac{a-b}{2}\right)^2\right]$$

$$-6\left[5\left(\frac{a+b}{2}\right)^2 + 2(a+b)c + 5\left(\frac{a-b}{2}\right)^2\right]^2$$

$$= 6.[7\left(5\left(\frac{a+b}{2}\right)^2 + 2(a+b)c\right)\left(\frac{a-b}{2}\right)^2 + \frac{5}{3}\left(\frac{a+b-2c}{2}\right)^2\left(\frac{a+b-2c}{2}\right)^2$$

$$+\frac{17}{3}\left(\frac{a+b-2c}{2}\right)^2\left(\frac{a-b}{2}\right)^2 - 25\left(\frac{a-b}{2}\right)^2]$$

$$\geq \frac{3}{2}\left(\frac{a-b}{2}\right)^2 (35(a+b)^2 - 25(a-b)^2) \geq 0.$$

We are done ,equality occurs if and only if $a = b = c$ or $a = b, c = 0$ and its permutation.

***4c)*** Let $a, b, c > 0$,prove that,

$$\left(\frac{b^2c + abc}{a^3 + abc} + 1\right)\left(\frac{c^2a + abc}{b^3 + abc} + 1\right)\left(\frac{a^2b + abc}{c^3 + abc} + 1\right) \geq 8$$

Solution, Let $\frac{b}{a} = x,\ \frac{a}{c} = y,\ \frac{c}{b} = z$. The Inequality becomes

$$(x^2 + 2x + y)(y^2 + 2y + z)(z^2 + 2z + x) \geq 8(x+y)(y+z)(z+x)$$

After expanding ,it becomes ,

$$4\sum x + 2\sum xy + 2\sum x^2y^2 + 2\sum x^3y + \sum x^3y^2 + \sum y^3x \geq 6 + 4\sum x^2y + 6\sum y^2x$$

$$\leftrightarrow 2\sum xy(x-1)^2 + \sum xy(y-1)^2 + \sum x(xy-1)^2 + 3\sum x + 2\sum x^2y^2 + 2\sum x^2y \geq 6 + 4\sum y^2x + \sum xy$$

Use

$$1, \sum xy(x-1)^2 + xy(y-1)^2 \geq 2xy(x+y-xy-1) = 2\sum x^2y + 2\sum xY^2 - 2\sum x^2y^2 - 2\sum xy$$

$$2, \sum yz(y-1)^2 + x(xy-1)^2 \geq 2\sum(y-1)(xy-1) = 2\sum y^2x + 6 - 2\sum x - 2\sum xy$$

It becomes ,

$$4\sum x^2z + \sum x \geq 5\sum xy$$

Let $x = \frac{a}{b},\ y = \frac{b}{c},\ z = \frac{c}{a}$, we get

$$4\sum a^3 + \sum a^2c \geq 45\sum a^2b$$

Or

$$3(\sum a^3 - \sum a^2b) + \sum a(a-b)^2 \geq 0$$

We are done ,equality occurs when $a = b = c$,we are done !

***4c)*** Let $a, b, c > 0$,prove that $(x^5 + y^5 + z^5)^2 \geq 3xyz(x^7 + y^7 + z^7)$

Solution, The Inequality equivalent

$$2\sum x^10 - \sum x^8(y^2 + z^2) + 3\sum x^8(y^2 + z^2) - 6\sum x^8yz \geq 2\sum(x^2y^8 + x^8y^2 - 2x^5y^5)$$

$$\Leftrightarrow \sum(x - y)^2(3z^8 + x^8 + y^8 + 2x^7y + 2xy^7 - 2x^5y^3 - 2x^3y^5 - 4x^4y^4) \geq 0$$

Since $x^7y + xy^7 - x^5y^3 - x^3y^5 = xy(x^2 + y^2)(x^2 - y^2)^2 \geq 0$
It suffice to prove that $\sum(x - y)^2(3z^8 + x^8 + y^8 - 4x^4y^4) \geq 0$
Without loss of generality ,assume $x \geq y \geq z$,.

$$S_x \geq 0,\ S_y = x^8 + z^8 + 3y^8 - 4x^4z^4 \geq (x^4 - 2z^4)^2 \geq 0$$

Thus ,

$$S_x(y - z)^2 + S_y(z - x)^2 + S_z(x - y)^2 \geq (x - y)^2(S_z + S_y)$$

$$= (x - y)^2(x^8 + 2y^8 + 2z^8 - 2x^4(y^4 + z^4) = (x - y)^2((x^4 - y^4 - z^4)^2 + (y^4 - z^4)^2) \geq 0$$

We are done!Equality occurs when $x = y = z$.

***4d)*** Let's $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that:

$$\sum \frac{3(a + b)}{a^2 + ab + b^2} \geq \frac{16(ab + ac + bc)}{(a + b)(a + b)(c + a)}$$

Solution,

$$\frac{1}{a^2 + ab + b^2} \geq \frac{ab + bc + ca}{(a + b)^2(a + b + c)^2}$$

Solution:

first ,we assume $a + b = 2z,\ b + c = 2x,\ c + a = 2y$

this ineq is equivalanteto :

$$\sum \frac{3z}{3z^2 + (x - y)^2} \geq \frac{2\sum xy - \sum x^2}{xyz}$$

Or

$$\sum(x - y)^2(1 - \frac{2xy}{3z^2 + (x - y)^2}) \geq 0$$

,assume $x \geq y \geq z$

Let $S_z = (x - y)^2\left(1 - \dfrac{2xy}{3z^2 + (x - y)^2}\right),\quad S_x = (y - z)^2\left(1 - \dfrac{2yz}{3x^2 + (y - z)^2}\right)$

$$S_y = (x - z)^2\left(1 - \frac{2xz}{3y^2 + (x - z)^2}\right)$$

it is easy to see that $S_x \geq 0$,

and $3y^2 + (x-z)^2 - 2xz \geq x^2 + 4z^2 - 4xz \geq 0$ ,from which we have $S_y \geq 0$
also $3z^2 + (x-y)^2 - 2xy + 3y^2 - 2xz + (x-z)^2 = x^2 + 4y^2 - 4xy + x^2 + 4z^2 - 4xz \geq 0$
it suffice to prove that:

$$\frac{(x-y)^2}{3z^2 + (x-y)^2} \leq \frac{(x-z)^2}{3y^2 + (x-z)^2} \Leftrightarrow (x-y)y \leq (x-z)z$$

or $x \leq y + z$
which is obviosly true by the assumed condition,so ,we have done

***4e)*** Let $a, b, c$ are three positive reals ,prove that ,

$$a^4 + b^4 + c^4 + 2(ab^3 + bc^3 + ca^3) \geq 2(a^3b + b^3c + c^3a) + a^2b^2 + b^2c^2 + c^2a^2$$

Solution, The inequality

$$\Leftrightarrow \sum a^2(a-b)^2 + 2\sum ab(b-c)^2 \geq \sum c^2(a-b)^2$$

Use the identity

$$\sum(a-b)^2(c^2 + ab) = \sum(a-b)^2(ac + bc)$$

It becomes

$$\sum(a-b)^2(a^2 + ac + ab - bc) \geq 0$$

Or

$$(a-b)^2(a+b)^2+2(b-a)(b-c)(b^2+bc+ba-ac)+(a-c)^2(b+c)^2 \geq 2|(b-a)(a-c)|(a+b)(b+c)+2(b-a)(b-c)(b^2+bc+ba-c$$

We are done !equality occurs when $a = b = c$ or $a = 2b, c = 0$ and it's permutation .

***4f)*** Let $a, b, c$ are three real numbers ,prove that ,

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a)$$

Solution,
WLOG, Assume $(b-a)(b-c) \leq 0$. Because RLH is $3(a^3b + b^3c + c^3a) \leq LHS$ so we only need prove the inequality in case

$$a^3b + b^3c + c^3a \geq ab^3 + bc^3 + ca^3 \Leftrightarrow c \geq b \geq a.$$

We can easy write

$$\sum(a-b)^2(2a^2 + c^2 - 2bc) \geq 0$$

$$\Leftrightarrow (a-b)^2(2a^2 + c^2 - 2bc) + (b-c)^2(2b^2 + a^2 - 2ca) + (c-a)^2(2c^2 + b^2 - 2ab)$$

$$= (a-b)^2(a^2+(a-b)^2+(a+b-c)^2)+(a-c)^2(b^2+(b-c)^2+(b+c-a)^2)+2(b-a)(a-c)(2b^2+a^2-2ca)$$

$$\geq 2|2(b-a)(a-c)|\sqrt{(a^2 + (a-b)^2 + (a+b-c)^2)(b^2 + (b-c)^2 + (b+c-a)^2)}+2(b-a)(a-c)(2b^2+a^2-2ca)$$

$$\geq 2|2(b-a)(a-c)|(|b(a+b-c)|+|a(b-c)|+|(a-b)(b+c-a)|)+2(b-a)(a-c)(2b^2+a^2-2ca)$$

$$= 2|2(b-a)(a-c)|(|ab+b^2-bc)|+|ab-ac)|+|a^2+b^2-2ab+bc-ac|)+2(b-a)(a-c)(2b^2+a^2-2ca)$$

$$\geq 2|2(b-a)(a-c)|(|2b^2+a^2-2ca|)+2(b-a)(a-c)(2b^2+a^2-2ca) \geq 0,$$

We are done ,equality occurs when $a=b=c$ or $a:b:c = sin(\frac{4\pi}{7})^2 : sin(\frac{2\pi}{7})^2 : sin(\frac{\pi}{7})^2$.

***4g)***Let $a,b,c \in R$ ,prove that,

$$4(a^4+b^4+c^4)+3(a^3b+b^3c+c^3a) \geq \frac{7}{27}(a+b+c)^4$$

Solution , Easy can rewwrite

$$\sum(a-b)^2(87(a+b)^2+27b^2+31c^2+106bc) = \sum(a-b)^2S_c \geq 0$$

Note that

$$S_c+S_a = 53(a+b+c)^2+34(a+c)^2+87(b+c)^2+31a^2+27b^2+5c^2$$

$$S_b+s_a = 53(a+b+c)^2+34(b+c)^2+87(a+c)^2+31b^2+27c^2+5a^2$$

Use Cauchy-Schwarz ,we can get

$$(S_a+S_c)(S_b+S_a)-S_a \geq 53a^2+75ab+(22+9\sqrt{2})b^2 \geq 0$$

We are done ,equality occurs when $a=b=c$.

***Problem 5.*** **(a (Vo Quoc Ba Can))** Let $a$, $b$, $c$ be positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{(b+c-a)(c+a-b)(a+b-c)}{2abc} \leq 2.$$

**(b(Nguyen Duy Tung))** Given a nonnegative real numbers $a,b$ and $c$ no two of wich are zero , prove that the following inequality holds

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{6(ab+bc+ca)}{(a+b+c)^2} \geq \frac{7}{2}$$

**(c)** Let $a,b,c$ are three positive reals ,prove that ,

$$\frac{b+c-a}{5a^2+4bc}+\frac{c+a-b}{5b^2+4ca}+\frac{a+b-c}{5c^2+4ab} \geq \frac{1}{a+b+c}$$

***Solution:***
(a) We have

$$1-\frac{(b+c-a)(c+a-b)(a+b-c)}{abc} = \sum\frac{a(a-b)(a-c)}{abc} = \sum\frac{(a-b)(a-c)}{bc}.$$

and

$$\sum\frac{2a}{b+c}-3 = \sum\frac{(a-b)(a-c)}{b+c}\left(\frac{1}{a+b}+\frac{1}{c+a}\right).$$

Thus it suffices to show that $\sum X_a(a-b)(a-c) \geq 0$
where

$$X_a = \frac{1}{bc}-\frac{1}{(b+c)(a+b)}-\frac{1}{(b+c)(a+c)}$$

we may assume that $a \geq b \geq c$

$$X_c = \frac{1}{ab}-\frac{1}{(b+c)(a+b)}-\frac{1}{(a+c)(a+c)} = \frac{1}{a+b}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{b+c}-\frac{1}{c+a}\right) \geq 0.$$

Thus it suffices to show that $X_a \geq X_b$ which reduces to $\dfrac{a-b}{abc} \geq \dfrac{a-b}{(a+b)(b+c)(c+a)}$ which is true because $(a+b+c) \leq \dfrac{(a+b)(b+c)(c+a)}{8} \leq (a+b)(b+c)(c+a)$.
The proof is completed.
(b) The inequality be equivalent to

$$\frac{b}{c+a}+\frac{c}{a+b}+\frac{a}{b+c}-\frac{1}{2} \geq \frac{2(a^2+b^2+c^2)}{(a+b+c)^2}$$

$$\Leftrightarrow \sum \frac{(a-b)^2}{2(a+c)(b+c)} \geq \sum \frac{(a-b)^2}{(a+b+c)^2}.$$

$$\Leftrightarrow \sum \frac{(a+b+c)^2-2(a+c)(b+c)}{2(a+b+c)^2(a+c)(b+c)} \geq 0.$$

Thus, we will have to prove after using Cauchy-Schwarz that

$$\sum (a-b)^2(a+b+c)^2 \geq 2\sum (a-b)^2(a+c)(b+c)$$

However, this can be also proved in a simple way, that is to notice that

$$2\sum (a-b)^2(a+c)(b+c) = 4\sum c(a+b)(a-b)^2$$

and $\sum (a-b)^2(a+b+c)^2 \geq 4\sum c(a+b)(a-b)^2$.
It is true because $(a+b+c) \geq 4(a+b)c$. So we have done.
The equality hold when $a=b=c$.
*c)* We have

$$\sum \frac{b+c-a}{5a^2+4bc}-\frac{1}{a+b+c} = (b-c)^2\left(\frac{5(a+b)-4c}{(5a^2+4bc)(5b^2+4ac)}\right)+\frac{c}{5a^2+4bc}+\frac{c}{5b^2+4ac}-\frac{6c^2-(a-b)^2}{(a+b+c)(5c^2+4ab)}$$

$$\geq \frac{c(6(a-b)^2+5a^2+5b^2+4(a+b)c)}{(5a^2+4bc)(5b^2+4ac)}-\frac{6c^2}{(a+b+c)(5c^2+4ab)}$$

It suffice to prove that

$$\frac{6(a-b)^2+5a^2+5b^2+4(a+b)c}{(5a^2+4bc)(5b^2+4ac)} \geq \frac{6c}{(a+b+c)(5c^2+4ab)}$$

Use the inequality

$$(5a^2+4bc)(5b^2+4ac) \geq \left(\frac{5a^2+5b^2+4ac+4bc}{2}\right)^2$$

And $(c-a)(c-b) \geq 0$ ,or $c^2+ab \geq ca+cb$

$$(a+b+c)(4a+4b+c)[5(a^2+b^2)+4(a+b)c+5(a-b)^2]-6\left(\frac{5(a^2+b^2)+4(a+b)c}{2}\right)^2$$

$$=\left[3(5\left(\frac{a+b}{2}\right)^2+2(a+b)c\right]+\left(\frac{a+b-2c}{2}\right)^2\left[10\left(\frac{a+b}{2}\right)^2+4(a+b)c+34\left(\frac{a-b}{2}\right)^2\right]$$

$$-6\left[5\left(\frac{a+b}{2}\right)^2+2(a+b)c+5\left(\frac{a-b}{2}\right)^2\right]^2$$

$$= 6.[7\left(5\left(\frac{a+b}{2}\right)^2+2(a+b)c\right)\left(\frac{a-b}{2}\right)^2+\frac{5}{3}\left(\frac{a+b-2c}{2}\right)^2\left(\frac{a+b-2c}{2}\right)^2$$

$$+\frac{17}{3}\left(\frac{a+b-2c}{2}\right)^2\left(\frac{a-b}{2}\right)^2-25\left(\frac{a-b}{2}\right)^2]$$

$$\geq \frac{3}{2}\left(\frac{a-b}{2}\right)^2(35(a+b)^2-25(a-b)^2)\geq 0.$$

We are done ,equality occurs if and only if $a=b=c$ or $a=b,c=0$ and its permutation.

***Problem 6.*** Let $a,b,c$ be nonnegative real numbers. Find the maximun of $k$ to such that inequality

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+k.\frac{ab+bc+ca}{a^2+b^2+c^2}\geq k+\frac{3}{2}.$$

***Solution:***

We have that equality $\sum\frac{a}{b+c}-\frac{3}{2}=\frac{1}{2}\sum\frac{(a-b)^2}{(a+c)(b+c)}$.

That inequality be rewrite that

$$\sum\frac{(a-b)^2}{(a+c)(b+c)}\geq k\sum\frac{(a-b)^2}{a^2+b^2+c^2}\Leftrightarrow\sum(a-b)^2\left(\frac{a^2+b^2+c^2}{(a+c)(b+c)}-k\right)\geq 0.$$

Let $b=c$ then $k$ need satifyl that condition with all $a,b\geq 0$

$$k\leq\frac{a^2+b^2+c^2}{(a+c)(b+c)}=\frac{a^2+2b^2}{2b(a+b)}.$$

We have easy that $\frac{a^2+2b^2}{2b(a+b)}\geq\frac{\sqrt{3}-1}{2}$.

We 'll prove that $k=\frac{\sqrt{3}-1}{2}$ is best constan. WLOG, assume $a\geq b\geq c$ so

$$S_a=\frac{a^2+b^2+c^2}{(a+c)(a+b)}-k, S_b=\frac{a^2+b^2+c^2}{(b+a)(b+c)}-k, S_c=\frac{a^2+b^2+c^2}{((c+a)(c+b)}-k.$$

We easy see that $S_c\geq S_b\geq S_a$ , also

$$S_b+S_a=\frac{(a^2+b^2+c^2)(a+b+2c)}{(a+b)(b+c)(c+a)}-2k.$$

Let $t=\frac{a+b}{2}$ we have

$$S_a+S_a\geq\frac{(2t^2+c^2)(2t+2c)}{2t(t+c)^2}-2k=\frac{2t^2+c^2}{t(t+c)}-2k.$$

The equality holds when $a=b=c$ or $a=b=\frac{\sqrt{3}+1}{2}c$ any cyclic permutation and $k=\frac{\sqrt{3}-1}{2}$.

***Problem 7(Vasile Cirtoaje)*** Let $a_1,a_2,...,a_n$ be nonnegative real numbers. Prove that

$$a_1^n+a_2^n+...+a_n^n+n(n-1)a_1a_2...a_n\geq a_1a_2...a_n(a_1+a_2+...+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right).$$

***Problem 8a(Nguyen Duy Tung).*** Let $a, b, c$ be the nonnegative real numbers. Prove that:

$$\frac{a^3+b^3}{c^2+ab}+\frac{b^3+c^3}{a^2+bc}+\frac{c^3+a^3}{b^2+ca}\geq 2\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)$$

***Solution:***

The inequality equivalent to

$$\sum\frac{(a+b)(b+c)(c+a)(a^3+b^3)}{c^2+ab}\geq 2[a^4+b^4+c^4+abc(a+b+c)+\sum a^3(b+c)].$$

$$\Leftrightarrow\sum(a+b)(a^3+b^3)(1+\frac{c(a+b)}{a^3+b^3}\geq 2[a^4+b^4+c^4+abc(a+b+c)+\sum a^3(b+c)].$$

$$\Leftrightarrow\sum\frac{c(a^3+b^3)(a+b)}{c^2+ab}\geq 2abc(a+b+c)+\sum a^3(b+c).$$

$$\Leftrightarrow\sum\left[\frac{c(a+b)(a^3+b^3)}{c^2+ab}+c^3(a+b)\right]\geq 2(a^2+b^2+c^2)(ab+bc+ca).$$

$$\Leftrightarrow\frac{a^4+b^4+c^4}{a^2+b^2+c^2}\left(\sum\frac{a(b+c)}{(a^2+bc)}\right)+abc\sum\frac{a+b}{c^2+ab}\geq 2(ab+bc+ca).$$

We have

$$\frac{a^4+b^4+c^4}{a^2+b^2+c^2}\left(\sum\frac{a(b+c)}{(a^2+bc)}\right)\geq\frac{(a^4+b^4+c^4)(a+b+c)^2}{a^2+b^2+c^2}\geq a^2+b^2+c^2.$$

and finally,we only need to prove that:

$$a^2+b^2+c^2+abc\sum\frac{b+c}{a^2+bc}\geq 2(ab+bc+ca)\Leftrightarrow a^2+b^2+c^2-ab-bc-ca\geq\sum ab\left(1-\frac{c(a+b)}{c^2+ab}\right).$$

Or $X(a-b)(a-c)+Y(b-a)(b-c++Z(c-a)(c-b)\geq 0.$

with $X=\dfrac{a^2}{a^2+bc}, Y=\dfrac{b^2}{b^2+ac}, Z=\dfrac{z^2}{z^2+ca}$. It is easy to see that $X\geq Y$ and $X, Y, Z\geq 0$.

So we have done.

We can see that with the cyclic inequality then we only assume a variable be $\min(a, b, c)$ , $\max(a, b, c)$ or a variable between any two variable with symmetry inequality so can't assume $a\geq b\geq c$ or equivalent it. So with cyclic inequality we must prove by two case $a\geq b\geq c$ or $c\geq b\geq a$. If we want to prove by SOS or VS then we must assessment by two case. That is a work really hard and take time. So we need find proof for it. So the question pose is how to resolved them? We 'll thinks that where are the cyclic inequality holds ? And we have the answer is in higher wages $a^2b+b^2c+c^2a, a^3b+b^3c+c^3a, \ldots$. They create permutation in inequality. And are $(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)$ ? Of course is no. They are $(ab^2+bc^2+ca^2)-(a^2b-b^2c-c^2a)=(a-b)(b-c)(c-a)$. Equivalent to it we have too $(a-b)^2(b-c)^2(c-a)^2$ and ...... And our idea is 'll establish a new form can solve cyclic inequality simply. That is $S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\geq S(a-b)(b-c)(c-a).(*)$. We 'll call it is Square And Cyclic (SAC).

Look at it we can see with cyclic inequality it can take easily to it.

And in cyclic inequality , we may assume that $b$ is number betwen two numbers $a$ and $c$ so we have two case are $a\geq b\geq c$ and $c\geq b\geq a$.

In case $a\geq b\geq c$ then $(a-b)(b-c)(c-a)\leq 0$ and so the inequality is true when $S\geq 0$

and $LHS = S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0$ then we can prove by SOS. And in case $c \geq b \geq a$ then $(a-b)(b-c)(c-a) \geq 0 \Leftrightarrow (ab^2 + bc^2 + ca^2 \geq a^2b + b^2c + c^2a$. With this case $(c \geq b \geq a)$ we have $(a-b)(b-c) \geq 0$ so

$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 = S_a(b-c)^2 + S_b(a-b+b-c)^2 + S_c(a-b)^2.$$

$$= (S_b + S_c)(a-b)^2 + (S_a + S_b)(b-c)^2 + 2S_b(a-b)(b-c)$$

$$\geq 2\sqrt{(S_a + S_b)(S_b + S_c)}(a-b)(b-c) + 2S_b(a-b)(b-c).$$

So to enough to prove

$$2\sqrt{(S_a + S_b)(S_b + S_c)} + 2S_b - S(c-a) \geq 0.$$

And we have $a - b = c - b - (c - a)$ and $b - c = b - a - (c - a)$ we have two way prove.
Continue we have $S_b(c-a)^2 = S_b(c-b+b-a)^2 \underset{AM-GM}{\geq} 4S_b(c-b)(b-a)$.
$S_a(b-c)^2 + S_c(a-b)^2 \geq 2\sqrt{S_a.S_c}.(b-a)(c-b)$.
So we need prove that $4S_b + 2\sqrt{S_a.S_c} \geq S(c-a)$.
Also we have too $S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 3\sqrt[3]{S_aS_bS_c(b-c)^2(c-a)^2(a-b)^2}$
And so we need prove $27S_aS_bS_c \geq S^3(a-b)(b-c)(c-a)$.

From arguments above we have that if one in 7 that case satisfy then $(*)$ true in case $c \geq b \geq a$. Seven standard are
1. $S_a + S_b \geq 0, S_b + S_c \geq 0, 2\sqrt{(S_a + S_b)(S_b + S_c)} + 2S_b - S(c-a) \geq 0$
2. $S_a + S_b \geq 0, S_a + S_c \geq 0, 2\sqrt{(S_a + S_b)(S_a + S_c)} - 2S_a - S(c-b) \geq 0$
3. $S_c + S_a \geq 0, S_c + S_b \geq 0, 2\sqrt{(S_c + S_a)(S_c + S_b)} - 2S_c - S(b-a) \geq 0$
4. $S_a \geq 0, S_c \geq 0, 2\sqrt{S_a.S_c} + 4S_b - S(c-a) \geq 0$
5. $S_a \geq 0, S_b \geq 0, S_c \geq 0, 2\sqrt{S_bS_c} - S(c-b) \geq 0$
6. $S_a \geq 0, S_b \geq 0, S_c \geq 0, 2\sqrt{S_aS_b} - S(b-a) \geq 0$
7. $S_a \geq 0, S_b \geq 0, S_c \geq 0, 27S_aS_bS_c - S^3(a-b)(b-c)(c-a) \geq 0$

The standard above are convenience to prove the inequality with $S_a, S_b, S_c$ are bulky. But when the inequality very strong then we 'll use different way is putting $c = a + x + y$ and $b = a + x$ with $x, y \geq 0$. This work 'll loss variable $a$ quickly. Because $c - a = x + y$ and $b - a = x$.
So how performance to have form $S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq S(a-b)(b-c)(c-a)$.

1. $ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a = (a-b)(b-c)(c-a)$.

2. $ab^2 + bc^2 + ca^2 - 3abc = \frac{1}{2}\left(ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a + \sum ab^2 + \sum a^2b - 6abc\right)$.

$$= \frac{1}{2}\left((a-b)(b-c)(c-a) + a(b-c)^2 + b(c-a)^2 + c(a-b)^2\right).$$

3. $\dfrac{a-b}{a+b} + \dfrac{b-c}{b+c} + \dfrac{c-a}{c+a} = \dfrac{-(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.

4. $\dfrac{a}{a+b} + \dfrac{b}{b+c} + \dfrac{c}{c+a} = \dfrac{1}{2}\left(\dfrac{a+b+a-b}{a+b} + \dfrac{b+c+b-c}{b+c} + \dfrac{c+a+c-a}{c+a}\right)$.

$$= \frac{1}{2}\left(3 + \frac{a-b}{a+b} + \frac{b-c}{b+c} + \frac{c-a}{c+a}\right) = \frac{1}{2}\left(3 - \frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}\right).$$

5. $ab^3+bc^3+ca^3-a^3b-b^3c-c^3a=(a+b+c)(a-b)(b-c)(c-a)$.

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Now we 'll enjoy some problem to see useful of this way.

***Problem 8b(Nguyen Huy Tung).*** Let $a,b,c$ be positive real numbers. Prove that

$$a^3+b^3+c^3+3abc.\frac{a^2b+b^2c+c^2a}{ab^2+bc^2+ca^2}\ge ab(a+b)+bc(b+c)+ca(c+a).$$

***Solution:***

WLOG , we may assume $b$ is number betwen to two numbers $a$ and $c$.

If $a\ge b\ge c$ then $a^2b+b^2c+c^2a\ge ab^2+bc^2+ca^2$. By Schur Inequality we have

$$a^3+b^3+c^3+3abc.\frac{a^2b+b^2c+c^2a}{ab^2+bc^2+ca^2}\ge a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a).$$

If $c\ge b\ge a$ then the inequality can rewrite

$$a^3+b^3+c^3-3abc+3abc.\left(\frac{a^2b+b^2c+c^2a}{ab^2+bc^2+ca^2}-1\right)\ge ab(a+b)+bc(b+c)+ca(c+a)-6abc$$

$$\Leftrightarrow \frac{1}{2}(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right)-\frac{3abc(a-b)(b-c)(c-a)}{ab^2+bc^2+ca^2}\ge a(b-c)^2+b(c-a)^2+c(a-b)^2.$$

$$\Leftrightarrow \frac{1}{2}(a+b-c)(a-b)^2+\frac{1}{2}(b+c-a)(b-c)^2+\frac{1}{2}(c+a-b)(c-a)^2\ge\frac{3abc(a-b)(b-c)(c-a)}{ab^2+bc^2+ca^2}$$

From criteria (1)

$$2\sqrt{ac}+c+a-b-\frac{3abc(c-a)}{ab^2+bc^2+ca^2}\ge 0$$

It equivalent to

$$2bc^2\left(\sqrt{ac}-a\right)+ab^2(c-b)+bc^2(c-b)+a^2c^2+a^2b^2+a^3c+2ab^2\sqrt{ac}+2ca^2\sqrt{ac}+2a^2bc\ge 0$$

This inequality is true because $c\ge b\ge a$.

The equality holds when $a=b=c$ or $(a,b,c)\sim(t,t,0)$.

***Problem 9(Nguyen Duy Tung, Nguyen Trong Tho).*** Let $a,b,c$ be positive real numbers. Prove that

$$\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2}\ge\frac{a+b+c}{3}.$$

***Solution:***

We have that

$$\sum\frac{a^3-ab^2}{2a^2+b^2}\ge 0\Leftrightarrow\sum(a^3-ab^2)(2b^2+c^2)(2c^2+a^2)\ge 0$$

$$\Leftrightarrow 3\sum a^3b^2c^2+2\sum a^3c^4+2\sum a^5b^2+\sum a^5c^2\ge 4\sum ab^4c^2+2\sum ab^2c^4+2\sum a^3b^4$$

$$\Leftrightarrow 2\sum\left(a^5b^2+a^3b^2c^2-2a^4b^2c\right)+\sum\left(a^5c^2+a^3b^2c^2-2a^4bc^2\right)\ge 2\left(\sum a^3b^4-a^3c^4\right)$$

$$\Leftrightarrow 2\sum a^3b^2(a-c)^2+\sum a^3c^2(a-b)^2\ge 2(a-b)(b-c)(c-a)\left(\sum a^2b^2+\sum a^2bc\right)$$

Now we need prove that $a = max\{a, b, c\}$
If $c < b$ we have $a \geq c \geq b$.
We 'll prove that

$$2a^3b^2(a-c)^2 + 2a^2c^3(c-b)^2 + a^3c^2(a-b)^2 \geq 2(a-c)(c-b)\left(a^3c^2 + a^3bc + a^3b^2\right)$$

Let two case.
Firstly case: $c - b \leq a - c$, we have

$$b^2(a-c)^2 + a^3c^2(a-b)^2 \geq 2a^3b^2(a-c)(c-b) + 4a^3c^2(a-c)(c-b)$$

And $a^3b^2 + 2a^3c^2 \geq a^3\left(c^2 + bc + b^2\right)$ so we have done.
Second case: $c - b > a - c$ we have too

$$2a^2c^3(c-b)^2 + a^3c^2(a-b)^2 \geq 2a^2c^3(a-c)(c-b) + 4a^3c^2(a-c)(c-b)$$

And $a^2c^3 + 2a^3c^2 - a^3\left(c^2 + bc + b^2\right) > a^2c^2b + a^3bc - a^3\left(bc + b^2\right) = a^3b(c-b) + a^2bc(c-a) \geq 0$
So we have done. The equality holds when $a = b = c$.
***Problem 10(Nguyen Duy Tung).*** Let $a, b, c$ be nonnegative real numbers. Prove that

$$\frac{4a}{a+b} + \frac{4b}{b+c} + \frac{4c}{c+a} + \frac{ab^2 + bc^2 + ca^2 + abc}{a^2b + b^2c + c^2a + abc} \geq 7$$

***Solution:***
We have that

$$2\left(3 - \frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}\right) + \left(\frac{ab^2 + bc^2 + ca^2 + abc}{a^2b + b^2c + c^2a + abc} - 1\right) \geq 6$$

$$\Leftrightarrow \frac{(a-b)(b-c)(c-a)}{a^2b + b^2c + c^2a + abc} - \frac{2(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)} \geq 0$$

$$\Leftrightarrow \frac{(a-b)(b-c)(c-a)\left[(a+b)(b+c)(c+a) - 2\left(a^2b + b^2c + c^2a + abc\right)\right]}{\left(a^2b + b^2c + c^2a + abc\right)(a+b)(b+c)(c+a)} \geq 0$$

$$\Leftrightarrow \frac{\left[(a-b)(b-c)(c-a)\right]^2}{\left(a^2b + b^2c + c^2a + abc\right)(a+b)(b+c)(c+a)} \geq 0.$$

The inequality is true.
The equality holds when $a = b = c$.
***Problem 11(Vasile Cirtoaje).*** Let $a, b, c$ be nonnegative real numbers. Prove that

$$a^3 + b^3 + c^3 + 2(a^2b + b^2c + c^2a) \geq 3(ab^2 + bc^2 + ca^2)$$

***Solution:***
WLOG , we may assume $b$ is number betwen to two numbers $a$ and $c$.
If $a \geq b \geq c$ then $2(a^2b + b^2c + c^2a) \geq 2(ab^2 + bc^2 + ca^2)$
And $a^3 + b^3 + c^3 \geq ab^2 + bc^2 + ca^2$ so the inequality is true.
If $c \geq b \geq a$ the inequality equivalent to

$$(a+b)(a-b)^2 + (b+c)(a-b)^2 + (c+a)(a-b)^2 \geq 5(a-b)(b-c)(c-a).$$

$$\Leftrightarrow (2a+b+c)(b-a)^2 + (2c+a+b)(c-b)^2 \geq (b-a)(c-b)(3c-7a)$$

Putting $c = a + x + y, b = a + x$, The inequality can rewrite that

$$x^2(4a + 2x + y) + y^2(4a + 3x + 2y) \geq xy(-4a + 3x + 3y)$$

Eliminated $a$ variable we have

$$x^2(2x + y) + y^2(3x + 2y) \geq xy(3x + 3y) \Leftrightarrow 2x^3 + 2y^3 \geq 2x^2y$$

The inequality is true because using AM-GM Inequality that $2x^3 + y^3 \geq 2x^2y$.
Same to that problem we have problem stonger

$$a^3 + b^3 + c^3 + 2(a^2b + b^2c + c^2a) \geq 3(ab^2 + bc^2 + ca^2) + \frac{3}{2}k(t - k)^2.$$

***Problem 12(Nguyen Duy Tung)*** Let $a, b, c$ be nonnegative real numbers. Prove that

$$4(a + b + c)^3 \geq 27\left(ab^2 + bc^2 + ca^2 + abc\right)$$

***Solution:***
WLOG , we may assume $b$ is number betwen to two numbers $a$ and $c$.
If $a \geq b \geq c$ then $ab^2 + bc^2 + ca^2 + abc \geq ab^2 + bc^2 + ca^2 + abc$ So

$$27\left(ab^2 + bc^2 + ca^2 + abc\right) \leq \frac{27}{2}\left(ab^2 + bc^2 + ca^2 + ab^2 + bc^2 + ca^2 + abc\right).$$

So we need prove $27\left(ab^2 + bc^2 + ca^2 + ab^2 + bc^2 + ca^2 + abc\right) \leq 8(a + b + c)^3$

$$\Leftrightarrow 8\left(a^3 + b^3 + c^3\right) \geq 3\left(ab^2 + bc^2 + ca^2 + ab^2 + bc^2 + ca^2\right) + 6abc.$$

Above inequality is true by AM-GM Inequality.
If $c \geq b \geq a$ the we rewrite the inequality

$$4\sum a^3 + 12\sum a^2b - 15\sum ab^2 - 3abc \geq 0$$

$$\Leftrightarrow 4\left(\sum a^3 - 3abc\right) - \frac{3}{2}\left[\sum(ab(a + b)) - 6abc\right] + \frac{27}{2}\left(\sum a^2b - \sum ab^2\right) \geq 0$$

$$\Leftrightarrow 2(a + b + c)\left[\sum(a - b)^2\right] - \frac{3}{2}\sum\left(a(b - c)^2\right) \geq \frac{27}{2}(c - b)(b - a)(c - a)$$

$$\Leftrightarrow (4b + 4c + a)(b - c)^2 + (4c + 4a + b)(c - a)^2 + (4a + 4b + c)(a - b)^2 \geq 27(c - b)(b - a)(c - a)$$

$$\Leftrightarrow (5a + 5b + 8c)(c - b)^2 + (8a + 5b + 5c)(b - a)^2 + 2(4a + b + 4c)(c - b)(b - a) \geq 27(c - b)(b - a)(c - a)$$

Putting $c = a + x + y, b = a + x$. The inequality equivalent to

$$y^2(18a + 8y + 13x) + x^2(18a + 5y + 10x) + 2(9a + 5x + 4y)xy \geq 27xy(x + y)$$

Eliminated $a$ variable we have

$$y^2(8y + 13x) + x^2(5y + 10x) + 2(5x + 4y)xy \geq 27xy(x + y) \Leftrightarrow 5x^3 + 4y^3 \geq 6x^2y + 3xy^2.$$

We have

$$2\left(x^3 + x^3 + y^3\right) \underset{AM-GM}{\geq} 6x^2y; x^3 + y^3 + y^3 \underset{AM-GM}{\geq} 6xy^2$$

So we have done.
Now we 'll strengthen the this inequality

$$\frac{1}{2}.18a\left(x^2+y^2+xy\right)\underset{AM-GM}{\geq}9a.\frac{3}{4}.(x+y)^2=\frac{27}{2}a(c-a)^2$$

So we have inequality stronger then it , $k=\min\{a,b,c\}$ and t = max{a, b, c} then

$$4(a+b+c)^3\geq 27\left(ab^2+bc^2+ca^2+abc\right)+\frac{27}{4}k(t-k)^2$$

Now you can practice with proposal problem.
***Problem 12(Nguyen Duy Tung).*** Let $a,b,c$ be nonnegative real numbers. Prove that

$$\frac{a^3+b^3+c^3}{3}\geq abc+\frac{1}{3}\sqrt{\frac{3\sqrt{3}+9}{\sqrt{3}-1}}|(a-b)(b-c)(c-a)|.$$

And $\frac{1}{3}\sqrt{\frac{3\sqrt{3}+9}{\sqrt{3}-1}}$ is better constan to the inequality true.
***Problem 13(Nguyen Duy Tung).*** Let $a,b,c$ be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that $ab^2+bc^2+ca^2\leq 2+abc$.
***Problem 14(Nguyen Duy Tung).*** Let $a,b,c$ be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that $(a+b+c)(a-b)(b-c)(c-a)\leq\frac{1}{4}$.
***Problem 15(Nguyen Duy Tung).*** Let $a,b,c$ be nonnegative real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3abc}{2\left(ab^2+bc^2+ca^2\right)}\geq 2.$$

With three variable inequality we have different nice method is SOS-Schur (SS) based on that equality

$$a^2+b^2+c^2-ab-bc-ca=(a^2+b^2-2ab)+(c^2-ca-cb+ab)=(a-b)^2+(a-c)(b-c).$$

And we know to SS: $K=M(a-b)^2+N(a-b)(a-c)$.
We know only need $c=\min(a,b,c)$ or $c=\max(a,b,c)$ then $a-c,b-c\geq 0$ and $M,N\geq 0$ so $K\geq 0$. Same too SOS and VS with all cyclic inequality or symmetry inequality we can too write it to form SS. Some useful equality

1. $a^2+b^2+c^2-ab-bc-ca=(a^2+b^2-2ab)+(c^2-ac-ab+ab)=(a-b)^2+(a-c)(b-c)$.
2. $a^3+b^3+c^3-3abc=(a+b+c)\left[(a-b)^2+(a-c)(b-c)\right]$.
3. $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{a}{b}+\frac{b}{a}-2+\frac{b}{c}+\frac{c}{a}-1-\frac{b}{a}=\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{ac}$.
4. $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\frac{(a+b)(a-b)^2}{ab}+\frac{(b+c)(a-c)(b-c)}{ac}$.
5. $\frac{a+kb}{a+kc}+\frac{b+kc}{b+ka}+\frac{c+ka}{c+kb}-3=\frac{k^2.(a-b)^2}{(c+ka)(c+kb)}+\frac{k(a-c)(b-c)[(k^2-k+1).a+(k-1)b+kc]}{(a+kb)(b+ka)(c+kb)}$.

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We 'll do some problem to the beautiful in SS.
***Problem 16(Nguyen Duy Tung).***Let $a,b,c$ be nonnegative real numbers.Prove that

$$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2}\geq\frac{5}{2}+\frac{4a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

***Solution:***
The inequality equivalent to

$$2\sum\left[(a^2+bc)(a^2+b^2)(a^2+c^2)\right]\geq (a^2+b^2)(b^2+c^2)(c^2+a^2)+8a^2b^2c^2.$$

$$\Leftrightarrow 2\sum a^6+2\sum b^3c^3+2abc\sum a^3+2abc\sum a^2(b+c)\geq 3\sum a^4(b^2+c^2)+12a^2b^2c^2.$$

WLOG, assume that $c=\min(a,b,c)$, we have $LHS-RHS=M(a-b)^2+N(a-b)(a-c)$.
with

$$M=2(a^4+b^4)+4ab(a^2+b^2)+a^2b^2+abc^2+(a+b)c^3+(2a^2b^2-a^2c^2-b^2c^2)$$

$+2c(a^2b+ab^2-a^2c-b^2c)\geq 0$ and

$$N=c[(3ab+2c^2)(a+b)+4abc+2c^3+(a^2b+ab^2-a^2c-b^2c)]\geq 0.$$

So we have done.
***Problem 9(Nguyen Duy Tung).*** Let $a,b,c$ be nonnegative real numbers.Prove that

$$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2}\geq\frac{5}{2}+\frac{4a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

***Solution:***
The inequality equivalent to

$$2\sum\left[(a^2+bc)(a^2+b^2)(a^2+c^2)\right]\geq (a^2+b^2)(b^2+c^2)(c^2+a^2)+8a^2b^2c^2.$$

$$\Leftrightarrow 2\sum a^6+2\sum b^3c^3+2abc\sum a^3+2abc\sum a^2(b+c)\geq 3\sum a^4(b^2+c^2)+12a^2b^2c^2.$$

WLOG, assume that $c=\min(a,b,c)$, we have $LHS-RHS=M(a-b)^2+N(a-b)(a-c)$.
with

$$M=2(a^4+b^4)+4ab(a^2+b^2)+a^2b^2+abc^2+(a+b)c^3+(2a^2b^2-a^2c^2-b^2c^2)+2c(a^2b+ab^2-a^2c-b^2c)\geq 0.$$

$$N=c[(3ab+2c^2)(a+b)+4abc+2c^3+(a^2b+ab^2-a^2c-b^2c)]\geq 0.$$

So we have done.
***Problem 10.*** Let $a,b$ and $c$ are positive numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+ac+bc}}$$

***Solution:***
Notice that if $a\geq b\geq c$ then

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\frac{(a-b)(a-c)(c-b)}{abc}\leq 0.$$

so it enough to consider the case $a\geq b\geq c$, we 'll prove

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\geq 9\left(\frac{a^2+b^2+c^2}{ab+ac+bc}\right).$$

Rewrite this inequality to $M(b-c)^2+N(a-b)(a-c)\geq 0$
With $M=\dfrac{2}{bc}+\dfrac{(b+c)^2}{b^2c^2}-\dfrac{9}{ab+bc+ca}$ and $N=\dfrac{2}{ac}+\dfrac{(a+b)(a+c)}{a^2b^2}-\dfrac{9}{ab+bc+ca}$.

We conclude that $M(b-c)^2+N(a-b)(a-c)\geq \dfrac{1}{2}(a-b)(a-c)(M+2N)\geq 0$

Now suppose that $b-c\leq a-b$ then $2b\leq a+c$. Certainly $M\geq 0$ and

$$N\geq \frac{2}{ac}+\frac{a+b+c}{ab^2}-\frac{9}{ab+bc+ca}\geq \frac{2}{ac}+\frac{3}{ab}-\frac{9}{ab+bc+ca}\geq \frac{(\sqrt{2}-\sqrt{3})^2}{ac+ab}-\frac{9}{ab+bc+ca}>0.$$

***Problem 11(Nguyen Duy Tung).*** Let $a,b,c$ be positive real numebrs . Prove that

$$\frac{a(b+c)}{b^2+c^2}+\frac{b(c+a)}{c^2+a^2}+\frac{c(a+b)}{a^2+b^2}\geq 2+\frac{8a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

***Solution:***

The inequality equivalent to

$$\sum[a(b+c)(a^2+b^2)(a^2+c^2)]\geq 2(a^2+b^2)(b^2+c^2)(c^2+a^2)+8a^2b^2c^2.$$

$$\Leftrightarrow \sum a^5(b+c)+2\sum b^2c^2+abc\sum a^2(b+c)\geq 2\sum a^4(b^2+c^2)+12a^2b^2c^2.$$

WLOG, assume that $c=\min(a,b,c)$. We have $LHS-RHS=M(a-b)^2+N(a-c)(b-c)\geq 0$. With $M=2(a^2+b^2+c^2)(a=b-c)c\geq 0$. And

$$N=(a^2+b^2+c^2)(a-b)^2+(a^3+b^3)c+(a+b)c^3+2c(a^2b+b^2c-a^2c-b^2c)\geq 0.$$

So we have done.

***Problem 12(Vo Quoc Ba Can).*** Let $a,b,c$ be positive real numbers . Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{8}{3}.\frac{ab+bc+ca}{a^2+b^2+c^2}\geq \frac{17}{3}.$$

***Solution:***

The inequality equivalent

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)+\frac{8}{3}.\left(\frac{ab+bc+ca}{a^2+b^2+c^2}-1\right)\geq 0.$$

$$\Leftrightarrow M(a-b)^2+N(a-b)(a-c)\geq 0.$$

With $M=\dfrac{1}{ab}-\dfrac{8}{3(a^2+b^2+c^2)}$ and $N=\dfrac{1}{ac}-\dfrac{8}{3(a^2+b^2+c^2)}$.

So if assume $c=\min(a,b,c)$ or $c=\max(a,b,c)$ then $M$ or $N\geq 0$ but don't prove $M,N\geq 0$. Asume $c=\min(a,b,c)$ then

$$3ac(a^2+b^2+c^2)N=3(a^2+b^2+c^2)-8ac\geq 3(a^2+2c^2)-8ac\geq 0.$$

So we have $N\geq 0$. Now,we have the question? What the condition then $M\geq 0$.
+) If $b-c\leq \dfrac{a-b}{4}$ then $c\geq \dfrac{5b-a}{4}$, we 'll prove that $M\geq 0$. Indeed we need prove

$$3(a^2+b^2+c^2)-8ab\geq 0.$$

If $a\geq 5b$ then the inequality is true, reverse we have

$$3(a^2+b^2=c^2)-8ab\geq 3a^2+3b^2+\frac{3(5b-a)^2}{16}-8ab=\frac{51a^2+123b^2-158ab}{16}\geq 0.$$

And this case we have $M \geq 0$ and $N \geq 0$ so we have done.
+) If $b - c \geq \dfrac{a-b}{4}$. In this case we easy see $a - c \geq \dfrac{5(a-b)}{4}$ so $(a-c)(b-c) \geq \dfrac{5}{16}(a-b)^2$.
Thus to prove originally we must prove that $M + \dfrac{5}{16}N \geq 0$ or $\dfrac{16}{ab} + \dfrac{5}{ac} \geq \dfrac{56}{a^2+b^2+c^2}$.
Apply AM-GM Inequality we have

$$\frac{16}{ab} + \frac{5}{ac} = \frac{8}{ab} + \frac{8}{ab} + \frac{5}{ac} \geq \frac{12\sqrt[3]{5}}{\sqrt[3]{a^3b^2c}}.$$

And

$$a^2 + b^2 + c^2 = 3\frac{a^2}{2} + 2\frac{b^2}{2} + c^2 \geq \frac{6}{\sqrt[6]{3^3.2^2}} = \frac{6}{\sqrt[6]{3^3.2^2}}\sqrt[3]{a^3b^2c} > \frac{14}{3\sqrt[3]{5}}\sqrt[3]{a^3b^2c}.$$

So $\dfrac{16}{ab} + \dfrac{5}{ac} \geq \dfrac{12\sqrt[3]{5}}{\sqrt[3]{a^3b^2c}} > \dfrac{12\sqrt[3]{5}}{\dfrac{3\sqrt[3]{5}}{14}(a^2+b^2+c^2)} = \dfrac{56}{a^2+b^2+c^2}$.

So we have done. The equality holds when $a = b = c$.

In this solution, we have a putting that in case 1 to $M \geq 0$ and in the case left we only need prove $M + kN \geq 0$ with $k$ is a positive real constand. So it can easy that inequality. Thus we can see that, Firstly if we assume $c = \min(a, b, c)$ or $c = \max(a, b, c)$ then $M$ or $N$ is nonnegative real numbers. Second with only in two numbers $M, N$ then if can't $M$ and $N$ are nonnegative numbers. So we must let min case to both $M$ and $N$ are nonnegative. And left case we 'll prove the inequality by easy than prove $M$ or $N$ nonegative real numbers. Now we 'll prove different problem to practice this way.

***Problem 13(Nguyen Duy Tung, Vo Quoc Ba Can).*** Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{6(a^2+b^2+c^2) - 3(ab+bc+ca)}{a+b+c}.$$

***Solution:*** WLOG, Assume $b$ is number between two numbers $a$ and $c$.
In case $c \geq b \geq a$ then: $\dfrac{a^2}{b} + \dfrac{b^2}{c} + \dfrac{c^2}{a} \geq \dfrac{a^2}{c} + \dfrac{c^2}{b} + \dfrac{b^2}{a}$.
Ineed, it equivalent to $\sum \dfrac{a^2}{b} - \sum \dfrac{a^2}{c} = \dfrac{a^3-b^3}{ab} + \dfrac{b^3-c^3}{bc} + \dfrac{c^3-a^3}{ca} \geq 0$.

$$\Leftrightarrow \frac{(c^3-b^3)(b-a)}{abc} + \frac{(b^3-a^3)(b-c)}{abc} \geq 0 \Leftrightarrow \frac{(c-b)(b-a)(b^2+c^2+bc+a^2+b^2+ab)}{abc} \geq 0.$$

It is true because $c \geq b \geq a$. So we need prove that

$$\frac{a^2}{c} + \frac{c^2}{b} + \frac{b^2}{a} \geq \frac{6(a^2+b^2+c^2) - 3(ab+bc+ca)}{a+b+c}.$$

Letting $a' = c, b' = c, c' = a$ and the inequality be equivalent to the inequality equivalent to original inequality. So we only need prove original inequality.

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} - (a+b+c) \geq \frac{6(a^2+b^2+c^2) - 3(ab+bc+ca)}{a+b+c} - (a+b+c).$$

$$\Leftrightarrow M(a-b)^2 + N(a-c)(b-c) \geq 0.$$

With $M = \dfrac{a+b}{ab} - \dfrac{5}{a+b+c}$ and $N = \dfrac{b+c}{ac} - \dfrac{5}{a+b+c}$.

We 'll prove the inequality by two case:
+) If $b-c \geq \dfrac{a-b}{2}$ and so $b \geq \dfrac{a+2c}{3}$ and $a-c \geq \dfrac{3(a-b)}{2}$ and thus $(a-c)(b-c) \geq \dfrac{3}{4}(a-b)^2$.

We must prove that $N \geq 0 \Leftrightarrow (b+c)(a+b+c) - 5ac \geq 0$. Indeed we have

$$(b+c)(a+b+c) - 5ac \geq \left(\frac{a+2c}{3} + c\right)\left(a + \frac{a+2c}{3} + c\right) - 5ac = \frac{(2a-5c)^2}{9} \geq 0.$$

And so we have $N(a-c)(b-c) \geq \dfrac{3}{4}N(a-b)^2$ so we enought to prove $M + \dfrac{3}{4}N \geq 0$.
+) If $b-c \leq \dfrac{a-b}{2} \Rightarrow c \geq \dfrac{3b-a}{2}$ and $a-b \geq \dfrac{2(a-c)}{3}$ and so $(a-b)^2 \geq \dfrac{4}{3}(a-c)(b-c)$.

We 'll prove that $M \geq 0 \Leftrightarrow (a+b)(a+b+c) - 5ab \geq 0$, Indeed

$$(a+b)(a+b+c) - 5ab \geq (a+b)\left(a+b+\frac{3b-a}{2}\right) - 5ab = \frac{(a-2b)^2+b^2}{2} > 0.$$

Thus $M(a-b)^2 \geq \frac{4}{3}M(a-c)(b-c)$. And we need prove that $\dfrac{4}{3}M + N \geq 0$ .

$$\Leftrightarrow 4\left(\frac{a+b}{ab} - \frac{5}{a+b+c}\right) + 3\left(\frac{b+c}{ac} - \frac{5}{a+b+c}\right) \geq 0.$$

Or $(3b^2 + 7bc + 4ca)(a+b+c) \geq 35abc$.
Apply AM-GM Inequality we have

$$3b^2 + \frac{7bc}{2} + \frac{7bc}{2} + 2ca + 2ca \geq 5\sqrt[5]{147a^2b^4c^4}.$$

And $\dfrac{a}{3} + \dfrac{a}{3} + \dfrac{a}{3} + b + c \geq \dfrac{5}{\sqrt[5]{27}}\sqrt[5]{a^3bc}$.
Multiply them we have

$$(3b^2 + 7bc + 4ca)(a+b+c) \geq \frac{25\sqrt[5]{147}}{\sqrt[5]{27}}abc > 35abc.$$

And so the inequality by true.
***Problem 14(Nguyen Duy Tung).*** Let $a, b, c$ be three side-lengths of a triangle. Prove that

$$2\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) \geq a + b + c + \frac{b^2}{a} + \frac{c^2}{b} + \frac{a^2}{c}.$$

***Solution:***
Clearly, this one is equivalent to

$$\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(c-a)^2}{a} \geq \frac{(a-b)(b-c)(c-a)(a+b+c)}{abc}.$$

$$\Leftrightarrow \sum_{cyc} ac(a-b)^2 \geq (a-b)(a-c)(b-c)(a+b+c).$$

The above form shows that we only need to prove it in case $a \geq b \geq c$ and $a = b+c$ (indeed, we only need to prove $\sum(a+c)(a-b)^2 \geq 3(a-b)(a-c)(b-c)$ applying the mixing variables method again, it remains to prove that $a(a-b)^2+b(b+a)^2+a^2b \geq 3ab(a-b)$ which is obvious). So we only need to prove the initial problem in case $(a,b,c)$ are three lengths of a trivial triangle when $a=b+c$. The inequality becomes

$$2\left((b+c)^3c+c^3b+b^2(a+b)\right) \geq 2bc(b+c)^2+(b+c)^3b+b^3c+c^2(b+c).$$

$$\Leftrightarrow b^4-2b^3c-b^2c^2+4bc^3+c^4 \geq 0.$$

Because of the homogeneity, we may assume $c=1$ and prove $f(b) \geq 0$ for

$$f(b)=b^4-2b^3+4b+1.$$

By derivative, it 's easy to prove this property.
This ends the proof.
***Problem 15(Vo Quoc Ba Can).*** Let $a,b,c$ be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{21(ab+bc+ca)}{(a+b+c)^2} \geq 10.$$

In SS method we have all symmetry and cyclic inequality we sure changes to

$$M(a-b)^2+N(a-c)(b-c)^2 \geq 0 \Leftrightarrow M(a-b)^2+N(a-b+b-c)(b-c) \geq 0.$$

$$\Leftrightarrow K=M(a-b)^2+N(a-b)(b-c)+N(b-c)^2 \geq 0.(*)$$

As we know, if we may assume that $c=\min(a,b,c)$ or $c=\max(a,b,c)$ then we 'll easy prove that $M$ or $N$ is nonnegative.Without loss of generality assume $M \geq 0$ and $c=\max(a,b,c)$.Because $M \geq 0$ then if $N \geq 0$ then we finished the solution. And in case $N \leq 0$:
+) If $b=c$ then $K=M(a-b)^2 \geq 0$. So we have finish solution.
+) If $b \neq c$ then we divided two hands of $(*)$ with $(b-c)^2$ we have

$$M\left(\frac{a-b}{b-c}\right)^2+N\left(\frac{a-b}{b-c}\right)+N \geq 0.(**)$$

Because $M \geq 0$ . So in case $N \geq 0$ then the inequality sure true.
And In case $N<0$. We have $(**)$ is true when $M \geq 0$ and $\Delta_{(**)} \geq 0$. Ineed $N^2-4MN \geq 0$ It is true because that $M \geq 0 \geq N$ so $N, N-4M \leq 0$. When we have $M$ and $\Delta_{(**)}$ are nonnegative real numbers then if $\dfrac{-N}{2M} \leq 0$ then the inequality is true with all $\dfrac{a-b}{b-c} \geq 0$. And because $c=\max(a,b,c)$ so $\dfrac{a-b}{b-c} \geq 0$ if and only if $c \geq b \geq a$. And so we only need prove original inequality with case $c \geq a \geq b$ (Left case.)
And then are some problem to you can practices:
***Problem 16(Nguyen Duy Tung)*** Let $a,b,c$ be positive real numbers. Prove that

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(a+c)}{a^2+ac+c^2}+\frac{c(a+b)}{a^2+ab+b^2} \geq 2+\frac{3[(a-b)(b-c)(c-a)]^2}{(a^2+ab+b^2)(b^2+bc+c^2)(a^2+ac+c^2)}$$

***Problem 17(Nguyen Duy Tung)*** Let $x \geq y \geq z > 0$ be positive real numbers. Prove that

$$\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y} \geq x^2+y^2+z^2+\frac{[(x-y)(y-z)(z-x)]^2}{xyz(x+y+z)}.$$

***Problem 18(Nguyen Duy Tung)*** Let $a, b, c$ be positive real numbers and $0 \leq k \leq 1$. Prove that

$$\frac{a^2+kbc}{b^2-bc+c^2}+\frac{b^2+kac}{a^2-ac+c^2}+\frac{c^2+kab}{a^2-ab+b^2} \geq 2+k+\frac{(1+2k)abc(a^3+b^3+c^3)}{3(a^2-ab+b^2)(b^2-bc+c^2)(a^2-ac+c^2)}$$

***Problem 18 (Ukraine 2006)*** For all positive real numbers $a, b$, and $c$. Prove that

$$3(a^3+b^3+c^3+abc) \geq 4(a^2b+b^2c+c^2a).$$

***Problem 19*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand to that inequality always true

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+k.\frac{ab+bc+ca}{a^2+b^2+c^2} \geq 9+k.$$

The answre is $k = 4\sqrt{2}$.

***Problem 20*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand to that inequality always true

$$\frac{bc}{b^2+c^2+ka^2}+\frac{ca}{c^2+a^2+kb^2}+\frac{ab}{a^2+b^2+kc^2} \leq \frac{3}{5}.$$

The answre is $k = 3$.

For this form inequality we know a method change variable. For three numbers $a, b, c$ we put $p = a+b+c$ , $q = ab+bc+ca$ and $r = abc$.
Then we have

1. $a^2+b^2+c^2 = p^2-2q$.
2. $a^3+b^3+c^3 = p^3-3pq+3r$.
3. $(a+b)(b+c)(c+a) = pq-r$
4. $a^4+b^4+c^4 = p^4-2q^2+4pr-4p^2q$.
5. $a^2b^2+b^2c^2+c^2a^2 = q^2-2pr$.
6. $a^2(b+c)+b^2(c+a)+c^2(a+b) = pq-3r$.
7. $a^3(b+c)+b^3(c+a)+c^3(a+b) = p^2q-2q-pr$.
8. $(a-b)^2(b-c)^2(c-a)^2 = p^2q^2+18pqr-27r^3-4q^3-4p^3r$.

.......................

The function $F(X) = AX^2+BX+C : \begin{cases} A \geq 0 \\ X_{min} = \dfrac{-B}{2A} \end{cases}$

(+) $F(X) \geq 0, \forall X \Leftrightarrow \Delta = B^2-4AC \leq 0$.

(+) $F(X) \geq 0, \forall X \geq 0 \Leftrightarrow \begin{cases} X_{min} \leq 0 \\ f(0) \geq 0 \end{cases} \left(\Leftrightarrow \begin{cases} B \geq 0 \\ C \leq 0 \end{cases}\right) Or \begin{cases} X_{min} \geq 0 \\ f(X_{min}) \geq 0 \end{cases} \left(\begin{cases} B \leq 0 \\ \Delta = B^2-4AC \geq 0 \end{cases}\right).$

***Problem 21*** Let $a, b, c$ be positive real numbers such that $a+b+c = 3$.Prove that

$$a^2b+b^2c+c^2a \leq 4$$

***Solution:*** The inequality equivalent to

$$2\sum_{cyc} a^2b \leq 8$$

$$\Leftrightarrow (\sum_{cyc} a^2b + \sum_{cyc} ab^2) + (\sum_{cyc} a^2b - \sum_{cyc} ab^2) \le 8$$

$$\Leftrightarrow \sum_{sym} a^2(b+c) + (a-b)(b-c)(c-a) \le 8$$

Then we need prove the inequality in case $(a-b)(b-c)(a-c) \ge 0$.

$$\sum_{sym} a^2(b+c) + \sqrt{(a-b)^2(b-c)^2(c-a)^2} \le 8$$

$$\Leftrightarrow (pq-3r) + \sqrt{(p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4q^3r)} \le 8$$

$$\Leftrightarrow (p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r) \le (8 - pq + 3r)^2$$

$$\Leftrightarrow 36r^2 + (4p^3 - 24pq + 48)r + 4q^3 - 16pq + 64 \ge 0$$

$$\Leftrightarrow 9r^2 + (p^3 - 6pq + 12)r + q^3 - 4pq + 16 \ge 0$$

We have $p = 3$ so

$$9r^2 + (39 - 18q)r + q^3 - 12q + 16 \ge 0$$

Putting $f(r) = 9r^2 + (39-18q)r + q^3 - 12q + 16$ for $r_{ct} = \dfrac{-39+18q}{18}$.

Let two case 1) $0 \le q \le \dfrac{39}{18} \Rightarrow r_{ct} \le 0$

$$f(0) = q^3 - 12q + 16 = (q+4)(q-2)^2 \ge 0$$

2) $\dfrac{39}{18} \le q \le 3 \Rightarrow r_{ct} \ge 0$

$$f(r_{ct}) = 24q^3 - 216q^2 + 648q - 630 \ge 0, \forall q \in \left[\frac{39}{18}; 3\right]$$

So $f(r) \ge 0 \forall r \ge 0 \Rightarrow$ we have done.

The equality holds when $a = b = c$.

***Problem 22*** Let $a, b, c$ be positive real numbers such that $a + b + c = 3$.Prove that

$$a^2b + b^2c + c^2a + 2(ab^2 + bc^2 + ca^2) \le 6\sqrt{3}$$

***Solution:*** The inequality equivalent to

$$\Leftrightarrow 2\sum_{cyc} a^2b + 4\sum_{cyc} ab^2 \le 12\sqrt{3}.$$

$$\Leftrightarrow 3\sum_{sym} a^2(b+c) + (\sum_{cyc} ab^2 - \sum_{cyc} a^2b) \le 12\sqrt{3}$$

$$\Leftrightarrow 3\sum_{cyc} a^2(b+c) + (a-b)(b-c)(c-a) \le 12\sqrt{3}$$

We only need prove inequality in case $(a-b)(b-c)(c-a) \ge 0$.

$$3\sum_{sym} a^2(b+c) + \sqrt{(a-b)^2(b-c)^2(c-a)^2} \le 12\sqrt{3}$$

$$\Leftrightarrow 3(pq - 3r) + \sqrt{p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r} \le 12\sqrt{3}$$

$$\Leftrightarrow p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r \le (12\sqrt{3} - 3pq + 9r)^2$$

$$\Leftrightarrow f(r) = 108r^2 + (4p^3 - 72pq + 216\sqrt{3})r + 4q^3 + 8p^2q^2 - 72\sqrt{3}pq + 432 \geq 0$$

Putting $r_{ct} = \dfrac{216q - 108 - 216\sqrt{3}}{108}$
Let two case
1) $0 \leq q \leq \dfrac{216\sqrt{3} + 108}{216} \Rightarrow r_{ct} \leq 0$

$$f(0) = 4(q + 12 + 6\sqrt{3})(q + 3 - \sqrt{3})^2 \geq 0.$$

2) $\dfrac{216\sqrt{3} + 108}{216} \leq q \leq 3 \Rightarrow r_{ct} \geq 0$

$$f(r_{ct}) = 4q^3 - 36q^2 + 108q + 81 - 108\sqrt{3} \geq 0, \forall q \in \left[\sqrt{3} + \frac{1}{2}; 3\right]$$

So $f(r) \geq r \forall r \geq 0$. The solution is end.
***Problem 23*** Let $a, b, c$ be positive real numbers such that $a + b + c = 3$.Prove that

$$k(a + b + c)^4 \geq (a^3b + b^3c + c^3a) + (a^2b^2 + b^2c^2 + c^2a^2) + abc(a + b + c).$$

***Solution:*** Let $a = 2, b = 1, c = 0 \Rightarrow k \geq \dfrac{4}{27}$.
We 'll prove $k \geq \dfrac{4}{27}$ is better constand.
The inequality equivalent to

$$\frac{4}{27}(a + b + c)^4 \geq \sum_{cyc} a^3b + \sum_{sym} b^2c^2 + abc\sum_{sym} a.$$

$$\Leftrightarrow \frac{8}{27}(a + b + c)^4 \geq (\sum_{cyc} a^3b + \sum_{cyc} ab^3) + 2\sum_{sym} b^2c^2 + (\sum_{cyc} a^3b - \sum_{cyc} ab^3) + 2abc(a + b + c)$$

$$\Leftrightarrow \frac{8}{27}(a+b+c)^4 \geq \sum_{sym} a^3(b+c) + 2\sum_{sym} b^2c^2 + (a+b+c)(a-b)(b-c)(a-c) + 2abc(a+b+c)$$

We only need prove inequality in case $(a - b)(b - c)(c - a) \geq 0$.

$$\Leftrightarrow \frac{8}{27}(a+b+c)^4 \geq p^2q - 2q^2 - pr + 2q^2 - 4pr + 2pr + p\sqrt{p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r}$$

$$\Leftrightarrow p^2(p^2q^2 + 18pqr - 27r^2 - 4q^3 - 4p^3r) \leq [\frac{8}{27}p^4 - p^2q + 3pr]^2$$

$$\Leftrightarrow 36p^2r^2 + (\frac{52}{9}p^5 - 24p^3q)r + \frac{64}{729}p^3 + 4p^2q^3 - \frac{16}{27}p^6q \geq 0$$

$$\Leftrightarrow 324p^2 + (1404 - 648q)r + 36q^3 - 432q + 576 \geq 0$$

Letting $f(r) = 36[9r^2 + (39 - 18q)r + q^3 - 12q + 16]$. Case 1: $0 \leq q \leq \dfrac{13}{6} \rightarrow 39 - 18q \geq 0$
We have $f(0) = 36(q + 4)(q - 2)^2 \geq 0$.
Case 2: $\dfrac{13}{6} \leq q \leq 3 \rightarrow \Delta = (39-18q)^2 - 4.9.(q^3 - 12q + 16) = -36q^3 + 324q^2 - 972q + 945 \leq 0$
$\forall q \in \left[\dfrac{13}{6}; 3\right]$ The solution is end.

***Problem 24*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + k\frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq 3 + k.$$

***Solution:*** We have

$$2\sum_{cyc}\frac{a}{b}=(\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a})+(\sum_{cyc}\frac{a}{b}-\sum_{cyc}\frac{b}{a})=\frac{\sum\limits_{sym}a^2(b+c)}{abc}+\frac{(a-b)(b-c)(c-a)}{abc}.$$

Thus, the inequality equivalent to

$$\sum_{cyc}\frac{a}{b}+2k\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 6+2k \Leftrightarrow \frac{\sum\limits_{sym}a^2(b+c)}{abc}+2k\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 6+2k+\frac{(a-b)(b-c)(a-c)}{abc}.$$

We only need prove inequality in case $(a-b)(b-c)(c-a)\geq 0$.

$$\Leftrightarrow \frac{pq-3r}{r}+\frac{2kq}{p^2-2q}\geq 6+2k+\frac{\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}}{r}$$

$$(p^2q^2+18pqr-27r^2-4q^3-4p^3q)(p^2-2q)^2\leq [(pq-3r)(p^2-2q)+2kqr-(6+2k)r(p^2-2q)]^2$$

Letting $f(r)=Ar^2+Br+C\geq 0$ (Assume $a+b+c=p=3$).

$$A=324k^2+36k^2q^2+216kq^2+2916k-3888q+432q^2+8748-216k^2q-1620kq$$

$$B=8748-432q^3+4320q^2-126361-72kq^3-972kq+540kq^2$$

And $C=16q^5-144q^4+324q^3$.

Case 1: $0\leq q\leq \dfrac{3(k+11-\sqrt{k^2+10k+49})}{2(k+6)}\Rightarrow B\geq 0$. We have $C\geq 0, A\geq 0\Rightarrow f(r)\geq 0$.

Case 2: $\dfrac{3(k+11-\sqrt{k^2+10k+49})}{2(k+6)}\leq q\leq 3$. We have

$$\Delta=B^2-4AC=-144(q-3)^2(2q-9)^2(48q^3+24kq^3+4k^2q^3-144kq^2-468q^2$$

$$-9k^2q^2+162kq+1296q-719)$$

So we have $k_{max}=3\sqrt[3]{4}-2$

***Problem 25*** Let $a,b,c$ be nonnegative real numbers. Find the beter constand to that inequality always true

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+k(a+b+c)\geq 3(k+1)\frac{a^2+b^2+c^2}{a+b+c}(6).$$

***Solution:*** We have

$$2\sum_{cyc}\frac{a^2}{b}=(\sum_{cyc}\frac{a^2}{b}+\sum_{cyc}\frac{b^2}{a})+(\sum_{cyc}\frac{a^2}{b}-\sum_{cyc}\frac{b^2}{a})=\frac{\sum\limits_{cyc}a^3(b+c)}{abc}+\frac{(a+b+c)(a-b)(b-c)(c-a)}{abc}$$

Thus, the inequality equivalent to

$$\Leftrightarrow 2(\sum_{cyc}\frac{a^2}{b}+2k(a+b+c)\geq 6(k+1)\frac{a^2+b^2+c^2}{a+b+c}$$

$$\Leftrightarrow \frac{\sum\limits_{cyc}a^3(b+c)}{abc}+2k(a+b+c)-6(k+1)\frac{a^2+b^2+c^2}{a+b+c}\geq \frac{(a+b+c)(a-b)(b-c)(a-c)}{abc}$$

We only need prove inequality in case $(a-b)(b-c)(c-a) \geq 0$.

$$\Leftrightarrow \frac{p^2q-2q^2-pr}{r}+2kp-6(k+1)\frac{p^2-2q}{p} \geq \frac{p\sqrt{p^2q^2+18pqr-27r^2-4q^3-4p^3r}}{r}$$

$\Leftrightarrow p^4(p^2q^2+18pqr-27r^2-4q^3-4p^3r) \leq [(p^2q-2q^2-pr)p+2kp^2r-6(k+1)r(p^2-2q)]^2$

$\Leftrightarrow f(r)=Ar^2+Br+C \geq 0$ (Assume $p=3$) For $A=288kq^2+144k^2q^2+1296k^2-1512q+4536k-2376kq-846k^2q+144q^2+6165$

$$B=8748-1944kq+1080kq^2-144kq^3-7776q+1404q^2-144q^3$$

And $C=36q^4$ . The equality $B=0$ have root $q \in [0,3]$.

$$q_0=\frac{1}{4(1+k)}(\sqrt[3]{M}+\frac{28k^2-100k-119}{\sqrt[3]{M}}+10k+3).$$

For $M=-1475-2382k-960k^2-80k^3+36\sqrt{N}+36k\sqrt{N}$
and $N=-12k^4+324k63-63k^2+2742k+2979$
Case 1: $0 \leq q \leq q_0 \Rightarrow B \geq 0; C \geq 0 \Rightarrow f(r) \geq 0$ (proved $A \geq 0$)
Case 2: $q_0 \leq q \leq 3$

$\Delta=B^2-4AC=-11644(q-3)^2(16q^3+16k^2q^3+32kq^3-252kq^2-189q^2-36k^2q^2+324kq+810q-729)$

So $\Rightarrow k_{max} \approx 1,5855400068$.

Thus, we can see this method is strong but it is unsimple, Need many computing, easy false. Same to it from form putting we know to PQR mathod. Letting $p=a+b+c, q=ab+bc+ca, r=abc$ We have equality too $.ab(a+b)+bc(b+c)+ca(c+a)=pq-3r$

$.(a+b)(b+c)(c+a)=pq-r$
$.ab(a^2+b^2)+bc^(b^2+c^2)+ca(c^2+a^2)=p^2q-2q^2-pr$
$.(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)=p^2+q$
$.a^2+b^2+c^2=p^2-2q$
$.a^3+b^3+c^3=p^3-3pq+3r$
$.a^4+b^4+c^4=p^4-4p^2q+2q^2+4pr$
$.a^2b^2+b^2c^2+c^2a^2=q^2-2pr$
$.a^3b^3+b^3c^3+c^3a^3=q^3-3pqr+3r^2$
$.a^4b^4+b^4c^4+c^4a^4=q^4-4pq^2r+2p^2r^2+4qr^2$

Letting $L=p^2q^2+18pqr-27r^2-4q^3-4p^3r$. Then $a^2b+b^2c+c^2a=\dfrac{pq-3r+/-\sqrt{L}}{2}$
For $(a-b)(b-c)(c-a)=\sqrt{L}$
We can see inequalities

$.p^2 \geq 3q$
$.p^3 \geq 27r$
$.q^2 \geq 3pr$
$.pq \geq 9r$
$.2p^3+9r \geq 7pq$
$.p^2q+3pr \geq 4q^2$
$.p^4+4q^2+6pr \geq 5p^2q$

The above result is certainly not enough, you can develop more equality, inequality between

three variables $p, q, r$. And it's important that I want to speak is two that inequality
$r \geq \dfrac{p(4q-p^2)}{9}$ $(i)$

$r \geq \dfrac{(4q-p^2)(p^2-q)}{6p}$ $(ii)$

However, in some cases it may be the quantity $4q - p^2$ can get negative values and positive values, so we often use

$$r \geq \max\left[0, \frac{p(4q-p^2)}{4}\right] \text{ or } r \geq \max\left[0, \frac{(4q-p^2)(p^2-q)}{6p}\right].$$

***Problem 25(Vo Thanh Van)*** Let $a, b, c$ be nonnegative real numbers. Prove that

$$\sqrt{\frac{(a+b)^3}{8ab(4a+4b+c)}} + \sqrt{\frac{(b+c)^3}{8bc(4b+4c+a)}} + \sqrt{\frac{(c+a)^3}{8ca(4c+4a+b)}} \geq 1$$

***Solution:*** Let $P = \sqrt{\dfrac{(a+b)^3}{8ab(4a+4b+c)}} + \sqrt{\dfrac{(b+c)^3}{8bc(4b+4c+a)}} + \sqrt{\dfrac{(c+a)^3}{8ca(4c+4a+b)}}$

$$Q = 8ab(4a+4b+c) + 8bc(4b+4c+a) + 8ca(4c+4a+b) = \sum 32ab(a+b) + 24abc$$

$$= 32(a+b+c)(ab+bc+ca) - 72abc$$

Apply Holder Inequality $P^2.Q \geq 8(a+b+c)^3$.
So we need prove that

$$8(a+b+c)^3 \geq Q \Leftrightarrow 8(a+b+c)^3 \geq 32(a+b+c)(ab+bc+ca) - 72abc$$

$$\Leftrightarrow (a+b+c)^3 \geq 4(a+b+c)(ab+bc+ca) - 9abc$$

It is Schur Inequality. The solution is end.
***Problem 25(APMO 2004)*** Let $a, b, c$ be nonnegative real numbers. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

***Solution:*** The equivalent to

$$a^2b^2c^2 + 2(a^2b^2 + b^2c^2 + c^2a^2) + 4(a^2+b^2+c^2) + 8 \geq 9(ab+bc+ca).$$

We have $a^2 + b^2 + c^2 \geq ab + bc + ca$

$$(a^2b^2+1) + (b^2c^2+1) + (c^2a^2+1) \geq 2(ab+bc+ca)$$

$$a^2b^2c^2 + 1 + 1 \geq 3\sqrt[3]{a^2b^2c^2} \geq \frac{9abc}{a+b+c} \geq 4(ab+bc+ca) - (a+b+c)^2$$

$$a^2b^2c^2 + 1 + 1 \geq 3\sqrt[3]{a^2b^2c^2} \geq \frac{9abc}{a+b+c} \geq 4(ab+bc+ca) - (a+b+c)^2$$

Apply above inequality we have

$$(a^2b^2c^2+2)+2(a^2b^2+b^2c^2+c^2a^2+3)+4(a^2+b^2+c^2) \geq 2\sum ab + 4\sum ab + 3\sum a^2 \geq 9\sum ab$$

The solution holds when $a=b=c$.

***Problem 26(Vo Thanh Van)*** Let $a,b,c$ be nonnegative real numbers. Prove that

$$\frac{a}{b^3+c^3}+\frac{b}{a^3+c^3}+\frac{c}{a^3+b^3}\geq\frac{18}{5(a^2+b^2+c^2)-ab-ac-bc}$$

***Solution:*** The equivalent to

$$\sum\frac{a(a+b+c)}{b^3+c^3}\geq\frac{18(a+b+c)}{5(a^2+b^2+c^2)-ab-bc-ca}$$

$$\Leftrightarrow\sum\frac{a^2}{b^3+c^3}+\frac{a}{b^2+c^2-bc}\geq\frac{18(a+b+c)}{5(a^2+b^2+c^2)-ab-bc-ca}$$

Apply Cauchy-Schwarz Inequality we have

$$i)\sum\frac{a^2}{b^3+c^3}\geq\frac{(a^2+b^2+c^2)^2}{\sum a^2(b^3+c^3)}$$

$$[ii)\sum\frac{a}{b^2+c^2-bc}\geq\frac{(a+b+c)^2}{\sum a(b^2+c^2-bc)}$$

Apply above inequality

$$\frac{(a^2+b^2+c^2)^2}{\sum a^2(b^3+c^3)}+\frac{(a+b+c)^2}{\sum a(b^2+c^2-bc)}\geq\frac{18(a+b+c)}{5(a^2+b^2+c^2)-ab-bc-ca}$$

Assume $a+b+c=1$ and $ab+bc+ca=q, abc=r\Rightarrow r\geq\max\left(0,\frac{(4q-1)(1-q)}{6}\right)$

We need prove that

$$\frac{(1-2q)^2}{q^2-(q+2)r}+\frac{1}{q-6r}\geq\frac{18}{5-11q}$$

Easy prove it by two case $1\geq 4q$ and $4q\geq 1$.

The equality holds when $a=b=c$ or $(a,b,c)\sim(t,t,0)$.

***Problem 27( Moldova TST 2005)*** Let $a,b,c$ be nonnegative real numbers such that $a^4+b^4+c^4=3$. Prove that

$$\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ca}\leq 1$$

***Solution:*** The equivalent to

$$49-8(ab+bc+ca)+(a+b+c)abc\leq 64-16(ab+bc+ca)+4(a+b+c)abc-a^2b^2c^2$$

$$\Leftrightarrow 16+3(a+b+c)abc\geq a^2b^2c^2+8(ab+bc+ca)$$

Apply Schur Inequality, we have

$$(a^3+b^3+c^3+3abc)(a+b+c)\geq(ab(a+b)+bc(b+c)+ca(c+a))(a+b+c)$$

$$\Leftrightarrow 3+3abc(a+b+c)\geq(ab+bc)^2+(bc+ca)^2+(ca+ab)^2$$

Apply AM-GM Inequality we have

$$(ab+bc)^2+(bc+ca)^2+(ca+ab)^2+12\geq 8(ab+bc+ca)$$

$$\Rightarrow 15+3abc(a+b+c)\geq 8(ab+bc+ca)$$

But we have too $1 \geq a^2b^2c^2$.So we have done.

***Problem 28(Vasile Cirtoaje)*** Let $a, b, c$ be nonnegative real numbers such that $ab+bc+ca = 3$. Prove that

$$a^3 + b^3 + c^3 + 7abc \geq 10$$

***Solution:*** Apply Schur inequality we have

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a)$$

$$\Leftrightarrow a^3 + b^3 + c^3 + 6abc \geq (ab+bc+ca)(a+b+c) = pq = 3p$$

Ans $r \geq \dfrac{p(4q-p^2)}{9} = \dfrac{p(12-p^2)}{9}$

We need prove that $3p + \dfrac{p(12-p^2)}{9} \geq 10 \Leftrightarrow \dfrac{(p-3)[(16-p^2)+3(4-p)+2]}{9} \geq 0.$

The solution is end. The equality holds when $a = b = c = 1$.

***Problem 29(Nguyen Phi Hung)*** Let $a, b, c$ be nonnegative real numbers such that $a^2 + b^2 + c^2 = 8$. Prove that

$$4(a+b+c-4) \leq abc$$

***Solution:*** From the condition we have $p^2 - 2q = 8$

Apply Schur Inequality we have

$$r \geq \frac{(4q-p^2)(p^2-q)}{6p} = \frac{(p^2-16)(p^2+8)}{12p}$$

So we need prove that

$$\frac{(p^2-16)(p^2+8)}{12p} \geq 4(p-4) \Leftrightarrow \frac{(p-4)^2(p^2+p-8)}{12p} \geq 0$$

So we have done. The equality holds $\Leftrightarrow a = b = 2, c = 0$ and any cyclic permutation.

***Problem 30*** Let $a, b, c > 0$ and $a+b+c = 1$.Prove that

$$\frac{\sqrt{a^2+abc}}{ab+c} + \frac{\sqrt{b^2+abc}}{bc+a} + \frac{\sqrt{c^2+abc}}{ca+b} \leq \frac{1}{2\sqrt{abc}}$$

***Solution:*** Changes $a, b, c$ to $p, q, r$ we have $r \leq \dfrac{q^2(1-q)}{2(2-3q)}$

Apply Cauchy-Schwarz Inequality we have

$$\left[\sum \frac{\sqrt{a^2+abc}}{(b+c)(b+a)}\right]^2 \leq \left[\sum \frac{a}{(a+b)(b+c)}\right]\left(\sum \frac{a+c}{b+c}\right) = \frac{\sum a^2 + \sum ab}{(a+b)(b+c)(c+a)}\left(\sum \frac{a+c}{b+c}\right)$$

We have

$$\sum \frac{a+c}{b+c} = \sum \frac{1}{b+c} - \sum \frac{b}{b+c} \leq \sum \frac{1}{b+c} - \frac{(a+b+c)^2}{\sum a^2 + \sum ab}$$

So we need prove that

$$\frac{\sum a^2 + \sum ab}{(a+b)(b+c)(c+a)}\left[\frac{1}{b+c} - \frac{1}{\sum a^2 + \sum ab}\right] \leq \frac{1}{4abc}$$

$$\Leftrightarrow \frac{1-q}{q-r}\left(\frac{1+q}{q-r} - \frac{1}{1-q}\right) \leq \frac{1}{4r} \Leftrightarrow \frac{4(1-q^2)}{q-r} - 4 \leq \frac{q-r}{r} \Leftrightarrow \frac{4(1-q^2)}{q-r} - \frac{q}{r} \leq 3$$

Using above inequality we have

$$LHS \leq \frac{4(1-q^2)}{q - \dfrac{q^2(1-q)}{2(2-3q)}} - \frac{q}{\dfrac{q^2(1-q)}{2(2-3q)}} = 3 - \frac{q(1-3q)(5-7q)}{(1-q)(4-7q+q^2)} \leq 3.$$

***Problem 31*** Let $a, b, c > 0$ .Prove that

$$\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 + \frac{10abc}{(a+b)(b+c)(c+a)} \geq 2$$

***Solution:*** We have

$$(a+b)(b+c)(c+a) \geq \frac{8}{9}(ab+bc+ca)(a+b+c) \geq \frac{8}{3}\sqrt[3]{a^2b^2c^2}(a+b+c)$$

Letting $x = \dfrac{2a}{b+c}, y = \dfrac{2b}{c+a}, z = \dfrac{2c}{a+b}$, we have $xy+yz+zx+xyz = 4$. Then the inequality equivalent

$$x^2 + y^2 + z^2 + 5xyz \geq 8$$

Take inequality to $p, q, r$,From the condition $q + r = 4$ and the inequality becomes

$$p^2 - 2q + 5r \geq 8 \Leftrightarrow p^2 - 7q + 12 \geq 0$$

If $4 \geq p$ using Schur Inequality

$$r \geq \frac{p(4q-p^2)}{9} \Rightarrow 4 \geq q + \frac{p(4q-p^2)}{9} \Leftrightarrow q \leq \frac{p^3+36}{4p+9} \Rightarrow p^2 - \frac{7(p^3+36)}{4p+9} + 12 \geq 0$$

$$\Leftrightarrow (p-3)(p^2-16) \leq 0$$

It is true because $4 \geq p \geq \sqrt{3q} \geq 3$ If $p \geq 4$ and $p^2 \geq 16 \geq 4q$, $p^2-2q+5r \geq p^2-2q \geq \dfrac{p^2}{2} \geq 8$. So the inequality is true. the equality holds when $x = y = z = 1$ or $x = y = 2, z = 0$ and any cyclic permutation.

***Problem 31*** Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Show that

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \geq \frac{9}{4}.$$

***Solution:*** We can rewrite inequality

$$4p^4q - 17p^2q^2 + 4q^3 + 34pqr - 9r^2 \geq 0$$

$$\Leftrightarrow pq(p^3 - 4pqr + 9r) + q(p^4 - 5p^2q + 4q^2 + 6pr) + r(pq - 9r) \geq 0$$

From Schur Inequality we have

$$p^3 \geq 4pqr + 9r \ , \ p^4 + 4q^2 + 6pr \geq 5p^2q \ , \ pq \geq 9r$$

So we have done.
The order to this section we 'll proposal.

***Problem 32*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + k.\frac{3abc}{ab^2+bc^2+ca^2} \geq k+1.$$

***Problem 33*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}+k.\frac{a^3b+b^3c+c^3a}{a^2b^2+b^2c^2+c^2a^2}\geq k+1.$$

***Problem 34*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{a^2+b^2+c^2}{(a+b+c)^2}+k.\frac{a^2b+b^2c+c^2a}{ab^2+bc^2+ca^2}\geq k+1.$$

$k_{max}=2.7775622......$

***Problem 35*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}+k.\frac{a^4b+b^4c+c^4a}{a^3b^2+b^3c^2+c^3a^2}\geq k+1.$$

$k_{max}\approx 0,89985223....$

***Problem 36*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+k(ab+bc+ca)\geq k(k+1)(a^2+b^2+c^2).$$

$k_{max}\approx 2.581412182....$

***Problem 37*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+k\geq (k+3)\sqrt[3]{\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2}.$$

$k_{max}\approx 0.3820494092....$

***Problem 38*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq k\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right).$$

***Problem 39*** Let $a, b, c$ be nonnegative real numbers. Find the condition necessary and sufficient of $k$ and $t$ to that inequality always true

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+k\geq (k+3)\left(\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}\right)^t.$$

***Problem 40*** Let $a, b, c$ be nonnegative real numbers. Find the beter constand $k$ to that inequality always true

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+k(a+b+c)\geq (3k+3)\sqrt{\frac{a^2+b^2+c^2}{3}}.$$

**Thanh Vân.**