歌西-舒瓦茲不等式的證明與推廣

Cauchy–Schwarz inequality

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一般而言，要理解或證明歌西-舒瓦茲不等式，最好的方法，是經由 n 維(度)向量空間來進行，不但省時省力，而且還可以利用向量空間中正交的投影性質，將此不等式在各領域中加以推廣，但可惜的是，在高中數學課程中關於向量部分只討論至三度空間，更高的維度只能請同學自行參考線性代數的相關書籍，底下所提出的說明，僅止於高中數學的範圍。

歌西-舒瓦茲不等式

$a_i \in \mathbb{R}$，$b_i \in \mathbb{R}$，$i = 1, 2, \cdots, n$

$\left(a_1b_1 + a_2b_2 + \cdots + a_nb_n\right)^2 \le \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$

等號成立 $\Leftrightarrow a_1 = tb_1$, $a_2 = tb_2$, $a_3 = tb_3$, $\cdots$, $a_n = tb_n$，$t \in \mathbb{R}$

證明：令 $\alpha = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}, \beta = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$

若 $a_1, a_2, a_3, \cdots, a_n$ 不全為 0 且 $b_1, b_2, b_3, \cdots, b_n$ 不全為 0，即 $\alpha \ne 0$ 且 $\beta \ne 0$

則由算幾不等式得：

$\frac{a_1^2}{\alpha^2} + \frac{b_1^2}{\beta^2} \ge 2 \cdot \sqrt{\frac{a_1^2}{\alpha^2} \times \frac{b_1^2}{\beta^2}} = 2 \cdot \frac{|a_1b_1|}{\alpha\beta}$ 且等號成立 $\Leftrightarrow \frac{a_1^2}{\alpha^2} = \frac{b_1^2}{\beta^2} \cdots\cdots(1)$

$\frac{a_2^2}{\alpha^2} + \frac{b_2^2}{\beta^2} \ge 2 \cdot \frac{|a_2b_2|}{\alpha\beta}$ 且等號成立 $\Leftrightarrow \frac{a_2^2}{\alpha^2} = \frac{b_2^2}{\beta^2} \cdots\cdots(2)$

$\frac{a_3^2}{\alpha^2} + \frac{b_3^2}{\beta^2} \ge 2 \cdot \frac{|a_3b_3|}{\alpha\beta}$ 且等號成立 $\Leftrightarrow \frac{a_3^2}{\alpha^2} = \frac{b_3^2}{\beta^2} \cdots\cdots(3)$

$\vdots$

$\frac{a_n^2}{\alpha^2} + \frac{b_n^2}{\beta^2} \ge 2 \cdot \frac{|a_nb_n|}{\alpha\beta}$ 且等號成立 $\Leftrightarrow \frac{a_n^2}{\alpha^2} = \frac{b_n^2}{\beta^2} \cdots\cdots(n)$

將上述不等式全數相加，即 $(1) + (2) + (3) + \cdots + (n)$ 得：

$$\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}{\alpha^2} + \frac{b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2}{\beta^2} \ge 2\left(\frac{|a_1b_1|}{\alpha\beta} + \frac{|a_2b_2|}{\alpha\beta} + \frac{|a_3b_3|}{\alpha\beta} + \cdots + \frac{|a_nb_n|}{\alpha\beta}\right) \cdots (*)$$

等號成立 $\Leftrightarrow \frac{a_1^2}{\alpha^2} = \frac{b_1^2}{\beta^2}$ 且 $\frac{a_2^2}{\alpha^2} = \frac{b_2^2}{\beta^2}$ 且 $\cdots$ $\frac{a_n^2}{\alpha^2} = \frac{b_n^2}{\beta^2}$

其中，因為 $\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}{\alpha^2} = 1$ 且 $\frac{b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2}{\beta^2} = 1$

故由(*)可知：

$$1 + 1 = 2 \ge 2\frac{|a_1b_1| + |a_2b_2| + a_3b_3| + \cdots + |a_nb_n|}{\alpha\beta}$$

$\Rightarrow \alpha\beta \ge |a_1b_1| + |a_2b_2| + a_3b_3| + \cdots + |a_nb_n|$

$\Rightarrow \alpha^2\beta^2 \ge \left(|a_1b_1| + |a_2b_2| + a_3b_3| + \cdots + |a_nb_n|\right)^2 \ge \left(\,|a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n|\,\right)^2$

注意：此時在上式第二個不等號之中若欲使等號成立須多加一個條件：

$a_1b_1, a_2b_2, a_3b_3, \cdots, a_nb_n$ 須皆同號

$\Rightarrow \alpha^2\beta^2 \ge \left(\,|\,a_1b_1+a_2b_2+a_3b_3+\cdots+a_nb_n\,|\,\right)^2 = \left(a_1b_1+a_2b_2+a_3b_3+\cdots+a_nb_n\right)^2$

$\Rightarrow \left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right) \ge \left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2$

等號成立 $\Leftrightarrow \dfrac{a_1^2}{\alpha^2}=\dfrac{b_1^2}{\beta^2}$ 且 $\dfrac{a_2^2}{\alpha^2}=\dfrac{b_2^2}{\beta^2}$ 且 $\cdots \dfrac{a_n^2}{\alpha^2}=\dfrac{b_n^2}{\beta^2}$ 且 $a_1b_1, a_2b_2, a_3b_3, \cdots, a_nb_n$ 須皆同號

此時關於等號成立的部分，分成以下兩種情況討論：

情況一：$a_1, a_2, a_3, \cdots, a_n, b_1, b_2, b_3, \cdots, b_n$ 全部不為0

$\Leftrightarrow \dfrac{a_1^2}{\alpha^2}=\dfrac{b_1^2}{\beta^2}$ 且 $\dfrac{a_2^2}{\alpha^2}=\dfrac{b_2^2}{\beta^2}$ 且 $\cdots \dfrac{a_n^2}{\alpha^2}=\dfrac{b_n^2}{\beta^2}$ 且 $a_1b_1, a_2b_2, a_3b_3, \cdots, a_nb_n$ 皆同號

$\Leftrightarrow \dfrac{a_1^2}{b_1^2}=\dfrac{a_2^2}{b_2^2}=\dfrac{a_3^2}{b_3^2}=\cdots=\dfrac{a_n^2}{b_n^2}=\dfrac{\alpha^2}{\beta^2}$ 且 $a_1b_1, a_2b_2, a_3b_3, \cdots, a_nb_n$ 皆同號

$\Leftrightarrow \left|\dfrac{a_1}{b_1}\right|=\left|\dfrac{a_2}{b_2}\right|=\left|\dfrac{a_3}{b_3}\right|=\cdots=\left|\dfrac{a_n}{b_n}\right|=\dfrac{\alpha}{\beta}$ 且 $a_1b_1, a_2b_2, a_3b_3, \cdots, a_nb_n$ 皆同號

$\Leftrightarrow \dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\dfrac{a_3}{b_3}=\cdots=\dfrac{a_n}{b_n}$ 並將此比值令為 t

$\Leftrightarrow a_1=tb_1$ ， $a_2=tb_2$ ， $a_3=tb_3$ ， $\cdots$ ， $a_n=tb_n$ ，$t\in\mathbb{R}$

情況二：$a_1, a_2, a_3, \cdots, a_n, b_1, b_2, b_3, \cdots, b_n$ 至少有一項為0

假設 $a_k=0$ ，則由算幾不等式等號成立時得知 $\dfrac{a_k^2}{\alpha^2}=\dfrac{b_k^2}{\beta^2}$ ，故 $b_k=0$

此時先將等於零之成對 a_k, b_k 排除後代入情況一之中討論亦可得：

$a_1=tb_1$ ， $a_2=tb_2$ ， $\cdots$ ， $0=a_k=tb_k=0$ ， $\cdots$ ， $a_n=tb_n$ ，$t\in\mathbb{R}$

又若 $a_1, a_2, a_3, \cdots, a_n, b_1, b_2, b_3, \cdots, b_n$ 全部為 0，則不等式顯然成立

故由上述討論可得知：

$\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2 \le \left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right)$

等號成立 $\Leftrightarrow a_1=tb_1$ ， $a_2=tb_2$ ， $a_3=tb_3$ ， $\cdots$ ， $a_n=tb_n$ ，$t\in\mathbb{R}$

Q.E.D.

推廣

利用上述的證明手法，可得到廣義歌西不等式：

設有 m 個**非負**實數數列：

$$a_{11}, a_{12}, a_{13}, \cdots, a_{1n}$$
$$a_{21}, a_{22}, a_{23}, \cdots, a_{2n}$$
$$a_{31}, a_{32}, a_{33}, \cdots, a_{3n}$$
$$\vdots$$
$$a_{m1}, a_{m2}, a_{m3}, \cdots, a_{mn}$$

則 $\left(a_{11}^m+a_{12}^m+a_{13}^m+\cdots+a_{1n}^m\right)\times\left(a_{21}^m+a_{22}^m+a_{23}^m+\cdots+a_{2n}^m\right)\times\cdots\times\left(a_{m1}^m+a_{m2}^m+a_{m3}^m+\cdots+a_{mn}^m\right)$

$\ge \left(a_{11}a_{21}a_{31}\cdots a_{m1}+a_{12}a_{22}a_{32}\cdots a_{m2}+\cdots+a_{1n}a_{2n}a_{3n}\cdots a_{mn}\right)^m$

等號成立 $\Leftrightarrow$ 數列 $(a_{11},a_{12},a_{13},\cdots,a_{1n}),(a_{21},a_{22},a_{23},\cdots,a_{2n}),\cdots,(a_{m1},a_{m2},a_{m3},\cdots,a_{mn})$ 兩兩成等比例

證明：令 $a_{11}^m+a_{12}^m+a_{13}^m+\cdots+a_{1n}^m=\alpha_1$

$$a_{21}^m + a_{22}^m + a_{23}^m + \cdots + a_{2n}^m = \alpha_2$$

$$\vdots$$

$$a_{m1}^m + a_{m2}^m + a_{m3}^m + \cdots + a_{mn}^m = \alpha_m$$

若$\alpha_k \neq 0$，$k = 1, 2, \cdots, m$，則由算幾不等式：

$$\frac{a_{11}^m}{\alpha_1} + \frac{a_{21}^m}{\alpha_2} + \frac{a_{31}^m}{\alpha_3} + \cdots + \frac{a_{m1}^m}{\alpha_m} \geq m \times \sqrt[m]{\frac{a_{11}^m}{\alpha_1} \times \frac{a_{21}^m}{\alpha_2} \times \frac{a_{31}^m}{\alpha_3} \times \cdots \times \frac{a_{m1}^m}{\alpha_m}} = m \times \frac{a_{11}a_{21}a_{31}\cdots a_{m1}}{\sqrt[m]{\alpha_1\alpha_2\alpha_3\cdots\alpha_m}}$$

$$\frac{a_{12}^m}{\alpha_1} + \frac{a_{22}^m}{\alpha_2} + \frac{a_{32}^m}{\alpha_3} + \cdots + \frac{a_{m2}^m}{\alpha_m} \geq m \times \sqrt[m]{\frac{a_{12}^m}{\alpha_1} \times \frac{a_{22}^m}{\alpha_2} \times \frac{a_{32}^m}{\alpha_3} \times \cdots \times \frac{a_{m2}^m}{\alpha_m}} = m \times \frac{a_{12}a_{22}a_{32}\cdots a_{m2}}{\sqrt[m]{\alpha_1\alpha_2\alpha_3\cdots\alpha_m}}$$

$$\frac{a_{13}^m}{\alpha_1} + \frac{a_{23}^m}{\alpha_2} + \frac{a_{33}^m}{\alpha_3} + \cdots + \frac{a_{m3}^m}{\alpha_m} \geq m \times \sqrt[m]{\frac{a_{13}^m}{\alpha_1} \times \frac{a_{23}^m}{\alpha_2} \times \frac{a_{33}^m}{\alpha_3} \times \cdots \times \frac{a_{m3}^m}{\alpha_m}} = m \times \frac{a_{13}a_{23}a_{33}\cdots a_{m3}}{\sqrt[m]{\alpha_1\alpha_2\alpha_3\cdots\alpha_m}}$$

$$\vdots$$

$$\frac{a_{1n}^m}{\alpha_1} + \frac{a_{2n}^m}{\alpha_2} + \frac{a_{3n}^m}{\alpha_3} + \cdots + \frac{a_{mn}^m}{\alpha_m} \geq m \times \sqrt[m]{\frac{a_{1n}^m}{\alpha_1} \times \frac{a_{2n}^m}{\alpha_2} \times \frac{a_{3n}^m}{\alpha_3} \times \cdots \times \frac{a_{mn}^m}{\alpha_m}} = m \times \frac{a_{1n}a_{2n}a_{3n}\cdots a_{mn}}{\sqrt[m]{\alpha_1\alpha_2\alpha_3\cdots\alpha_m}}$$

將上述不等式全部相加得：

$$m \geq m \times \frac{1}{\sqrt[m]{\alpha_1\alpha_2\alpha_3\cdots\alpha_m}} \times \left(a_{11}a_{21}a_{31}\cdots a_{m1} + a_{12}a_{22}a_{32}\cdots a_{m2} + \cdots + a_{1n}a_{2n}a_{3n}\cdots a_{mn}\right)$$

$$\Rightarrow \sqrt[m]{\alpha_1\alpha_2\alpha_3\cdots\alpha_m} \geq \left(a_{11}a_{21}a_{31}\cdots a_{m1} + a_{12}a_{22}a_{32}\cdots a_{m2} + \cdots + a_{1n}a_{2n}a_{3n}\cdots a_{mn}\right)$$

$$\Rightarrow \alpha_1\alpha_2\alpha_3\cdots\alpha_m \geq \left(a_{11}a_{21}a_{31}\cdots a_{m1} + a_{12}a_{22}a_{32}\cdots a_{m2} + \cdots + a_{1n}a_{2n}a_{3n}\cdots a_{mn}\right)^m$$

即$\left(a_{11}^m + a_{12}^m + a_{13}^m + \cdots + a_{1n}^m\right) \times \left(a_{21}^m + a_{22}^m + a_{23}^m + \cdots + a_{2n}^m\right) \times \cdots \times \left(a_{m1}^m + a_{m2}^m + a_{m3}^m + \cdots + a_{mn}^m\right)$

$$\geq \left(a_{11}a_{21}a_{31}\cdots a_{m1} + a_{12}a_{22}a_{32}\cdots a_{m2} + \cdots + a_{1n}a_{2n}a_{3n}\cdots a_{mn}\right)^m$$

又等號成立時，由算幾不等式：

$\Leftrightarrow \frac{a_{11}^m}{\alpha_1} = \frac{a_{21}^m}{\alpha_2} = \frac{a_{31}^m}{\alpha_3} = \cdots = \frac{a_{m1}^m}{\alpha_m}$ 且 $\frac{a_{12}^m}{\alpha_1} = \frac{a_{22}^m}{\alpha_2} = \frac{a_{32}^m}{\alpha_3} = \cdots = \frac{a_{m2}^m}{\alpha_m}$ 且……且 $\frac{a_{1n}^m}{\alpha_1} = \frac{a_{2n}^m}{\alpha_2} = \frac{a_{3n}^m}{\alpha_3} = \cdots = \frac{a_{mn}^m}{\alpha_m}$

$\Leftrightarrow n$ 個數列 $(a_{11}, a_{21}, a_{31}, \cdots, a_{m1}), (a_{12}, a_{22}, a_{32}, \cdots, a_{m2}), \cdots, (a_{1n}, a_{2n}, a_{3n}, \cdots, a_{mn})$ 兩兩成等比例

$\Leftrightarrow m$ 個數列 $(a_{11}, a_{12}, a_{13}, \cdots, a_{1n}), (a_{21}, a_{22}, a_{23}, \cdots, a_{2n}), \cdots, (a_{m1}, a_{m2}, a_{m3}, \cdots, a_{mn})$ 兩兩成等比例

又若$\alpha_k = 0$，$1 \leq k \leq m$，則$a_{k1} = a_{k2} = \cdots = a_{kn} = 0$，故不等式為$0 \geq 0$顯然成立

Q.E.D.

使用範例

(1) $\frac{4}{\sin\theta} + \frac{9}{\cos\theta}$之最小值為$\sqrt{\left(\sqrt[3]{4^2} + \sqrt[3]{9^2}\right)^3}$，其中設$0 < \theta < \frac{\pi}{2}$

解：$\left(\frac{4}{\sin\theta} + \frac{9}{\cos\theta}\right)^2 = \left(\frac{4}{\sin\theta} + \frac{9}{\cos\theta}\right)^2 \times 1 = \left(\frac{4}{\sin\theta} + \frac{9}{\cos\theta}\right)^2 \times \left(\sin^2\theta + \cos^2\theta\right)$

$$= \left(\left(\sqrt[3]{\frac{4}{\sin\theta}}\right)^3 + \left(\sqrt[3]{\frac{9}{\cos\theta}}\right)^3\right) \times \left(\left(\sqrt[3]{\frac{4}{\sin\theta}}\right)^3 + \left(\sqrt[3]{\frac{9}{\cos\theta}}\right)^3\right) \times \left(\left(\sqrt[3]{\sin^2\theta}\right)^3 + \left(\sqrt[3]{\cos^2\theta}\right)^3\right)$$

$$\geq \left(\left(\sqrt[3]{\frac{4}{\sin\theta}}\right)^3 \times \left(\sqrt[3]{\frac{4}{\sin\theta}}\right)^3 \times \left(\sqrt[3]{\sin^2\theta}\right)^3 + \left(\sqrt[3]{\frac{9}{\cos\theta}}\right)^3 \times \left(\sqrt[3]{\frac{9}{\cos\theta}}\right)^3 \times \left(\sqrt[3]{\cos^2\theta}\right)^3\right)^3$$

$$=\left(\sqrt[3]{4^2}+\sqrt[3]{9^2}\right)^3$$

$$\Rightarrow \frac{4}{\sin\theta}+\frac{9}{\cos\theta}\geq\sqrt{\left(\sqrt[3]{4^2}+\sqrt[3]{9^2}\right)^3}$$

等號成立 $\Leftrightarrow \sqrt[3]{\frac{4}{\sin\theta}}:\sqrt[3]{\frac{9}{\cos\theta}}=\sqrt[3]{\sin^2\theta}:\sqrt[3]{\cos^2\theta}\Rightarrow \sin\theta:\cos\theta=\sqrt[3]{4}:\sqrt[3]{9}$