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Source: *The Mathematical Gazette*, Vol. 49, No. 367 (Feb., 1965), pp. 66-69
Published by: The Mathematical Association
Stable URL: https://www.jstor.org/stable/3614241

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ABSTRACT

Goodner [1] has studied the equations to conic sections in the elliptic plane by following the methods of Euclidean geometry and using Weierstrassian co-ordinates. In the present note the author has investigated several properties of the elliptic circles from the same analytic standpoint and has shown that some of the well-known properties of the Euclidean circles hold as well for the elliptic circles.

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TWO PROBLEMS ON CONVEX FUNCTIONS

BY A. S. BESICOVITCH and ROY O. DAVIES

Given a positive continuous increasing function $f(x)$, $0 \leq x < 1$, approximations by a convex* function $g(x)$ from below and from above are considered. The approximation is measured by the ratio of the integrals $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$.

THEOREM 1. *There always exists a convex function* $g(x) \leq f(x)$ *such that*

$$\int_0^1 g(x)\,dx \geq \frac{1}{2}\int_0^1 f(x)\,dx. \tag{1}$$

Proof. Take the convex hull* of the curve $y = f(x)$, and let the curve $y = g(x)$ be the boundary of the hull from below. Obviously $g(x)$ is convex and also is $\leq f(x)$. If $f(x)$ is itself convex then $g(x)$ coincides with $f(x)$ and the Theorem is true. If not then $f(x) > g(x)$ on a set of open intervals. Let (x', x'') be one of them. We have

* See the Definitions at the end.

$g(x') = f(x')$, $g(x'') = f(x'')$, and in (x', x'') $g(x)$ is linear and $f(x) \leq f(x'')$. Hence

$$\int_{x'}^{x''} g(x)\,dx = \frac{1}{2}\{f(x') + f(x'')\}(x'' - x'),$$

$$\int_{x'}^{x''} f(x)\,dx < f(x'')(x'' - x'),$$

and

$$\int_{x'}^{x''} g(x)\,dx > \frac{1}{2}\int_{x'}^{x''} f(x)\,dx;$$

from which the Theorem follows.

The proof shows that strict inequality is always obtainable in (1), unless $\int_0^1 f(x)\,dx = \infty$; considering $f(x) = x^{1/n}$ for large n, we see however that the bound $\frac{1}{2}$ can not be improved.

THEOREM 2. *There always exists a convex function $g(x) \geqslant f(x)$ such that*

$$\int_0^1 g(x)\,dx \leq 2\int_0^1 f(x)\,dx. \tag{2}$$

Proof. Define $\varphi_0(x)$ to be the constant function $f(\frac{1}{2})$; thus $\varphi_0(x)$ is convex. Denote by R_0 the rectangle of which three vertices are at the points $(\frac{1}{2}, 0)$, $(\frac{1}{2}, f(\frac{1}{2}))$, and $(1, 0)$. Then

$$\int_0^1 \varphi_0(x)\,dx = 2(\text{area } R_0),$$

and throughout $(0, \frac{1}{2})$

$$\varphi_0(x) \geq f(x). \tag{3}$$

Denote by b_0 the largest number ≤ 1 such that (3) holds throughout $(0, b_0)$; thus $\frac{1}{2} \leq b_0 \leq 1$. If $b_0 = 1$, we take $g(x) = \varphi_0(x)$.

If $b_0 < 1$, write $1 - b_0 = 3l_0$ and let c_1 denote the largest value of x in $[b_0, b_0 + l_0]$ for which $\{f(x) - f(b_0)\}/\{x - (b_0 - l_0)\}$ attains its maximum; we have $b_0 < c_1 \leq b_0 + l_0$; $0 < 2c_1 - 1 \leq b_0 - l_0$. Define $\varphi_1(x)$ to be the function which is 0 in $[0, 2c_1 - 1]$ and linear in $[2c_1 - 1, 1]$, with $\varphi_1(c_1) = f(c_1) - f(b_0)$; then $\varphi_1(x)$ is convex. Denote by R_1 the rectangle of which three vertices are at the points $(c_1, f(b_0))$, $(c_1, f(c_1))$, and $(1, f(b_0))$. Then

$$\int_0^1 \varphi_1(x)\,dx = 2(\text{area } R_1),$$

and we shall now prove that throughout $(0, c_1)$

$$\varphi_0(x) + \varphi_1(x) \geq f(x). \tag{4}$$

This follows from (3) in $(0, b_0)$, while in (b_0, c_1) we have $\varphi_0(x) \geq f(b_0)$ and

$$\begin{aligned}\varphi_1(x) &= \frac{x-(2c_1-1)}{1-c_1}\{f(c_1)-f(b_0)\} \\ &\geq \frac{x-(2c_1-1)}{1-c_1}\cdot\frac{f(x)-f(b_0)}{x-(b_0-l_0)}\cdot\{c_1-(b_0-l_0)\} \\ &= \frac{x-(2c_1-1)}{x-(b_0-l_0)}\cdot\frac{c_1-(b_0-l_0)}{c_1-(2c_1-1)}\cdot\{f(x)-f(b_0)\} \\ &\geq f(x)-f(b_0),\end{aligned}$$

from which (4) follows. Denote by b_1 the largest number ≤ 1 such that (4) holds throughout $(0, b_1)$; thus $c_1 \leq b_1 \leq 1$. Since $\varphi_0(x) + \varphi_1(x)$ is the sum of convex functions, it is itself convex, and if $b_1 = 1$ we take $g(x) = \varphi_0(x) + \varphi_1(x)$.

If $b_1 < 1$, write $1 - b_1 = 3l_1$ and let c_2 denote the largest value of x in $[b_1, b_1 + l_1]$ for which $\{f(x) - f(b_1)\}/\{x - (b_1 - l_1)\}$ attains its maximum; we have $b_1 < c_2 \leq b_1 + l_1$; $0 < 2c_2 - 1 \leq b_1 - l_1$. Define $\varphi_2(x)$ to be the function which is 0 in $[0, 2c_2 - 1]$ and linear in $[2c_2 - 1, 1]$, with $\varphi_2(c_2) = f(c_2) - f(b_1)$; then $\varphi_2(x)$ is convex. Denote by R_2 the rectangle of which three vertices are at the points $(c_2, f(b_1))$, $c_2, f(c_2))$, and $(1, f(b_1))$. Then

$$\int_0^1 \varphi_2(x)\, dx = 2(\text{area } R_2),$$

and throughout $(0, c_2)$

$$\varphi_0(x) + \varphi_1(x) + \varphi_2(x) \geq f(x). \tag{5}$$

This follows from (4) in $(0, b_1)$, while in (b_1, c_2) we have $\varphi_0(x) + \varphi_1(x) \geq f(b_1)$ and by the same argument as before $\varphi_2(x) \geq f(x) - f(b_1)$. Denote by b_2 the largest number ≤ 1 such that (5) holds throughout $(0, b_2)$. As before $\varphi_0(x) + \varphi_1(x) + \varphi_2(x)$ is convex, and if $b_2 = 1$ we take $g(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x)$; if not, we continue the construction, possibly transfinitely.

The construction must terminate after denumerably many steps, and from (3), (4), ... it is clear that the resulting convex function $g(x) = \varphi_0(x) + \varphi_1(x) + \ldots$ will satisfy $g(x) \geq f(x)$ throughout $(0, 1)$. On the other hand the rectangles R_0, R_1, ... are non-overlapping and all contained in the area between the curve $y = f(x)$ and the x-axis. Consequently

$$\begin{aligned}\int_0^1 g(x)\, dx &= \int_0^1 \varphi_0(x)\, dx + \int_0^1 \varphi_1(x)\, dx + \ldots \\ &= 2\{(\text{area } R_0) + (\text{area } R_1) + \ldots\} \\ &\leq 2\int_0^1 f(x)\, dx,\end{aligned}$$

and the Theorem is proved.

The proof shows that strict inequality is always obtainable in (2), unless $\int_0^1 f(x)\,dx = \infty$; considering $f(x) = 1 + (2x - 1)^{1/(2n+1)}$, for large n, we see however that the bound 2 can not be improved.

DEFINITIONS

A function $g(x)$ is called *convex* if every arc AC of the curve $y = g(x)$ lies below, or coincides with, the chord AC. In analytic language:

$$g(b) \leq \frac{c-b}{c-a} g(a) + \frac{b-a}{c-a} g(c) \quad \text{whenever } a < b < c.$$

A point set is called *convex* if whenever two points P, Q belong to the set so does every point of the (straight line) segment PQ. (The boundary of every plane convex set is a convex curve: roughly speaking, this is a curve without dents.) The *convex hull* H of a plane set E is the common part of all convex sets containing E; it is easy to see that H is itself convex, and is thus the smallest convex set containing E. The set H can also be defined as $\varphi(\varphi(E))$, where (for any set X) $\varphi(X)$ denotes the union of X and the set of all points lying on segments joining points of X.

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PARTICULAR INTEGRALS OF LINEAR DIFFERENTIAL EQUATIONS

BY A. G. MACKIE

In elementary courses on differential equations the standard method of obtaining particular integrals is probably that of the variation of parameters or, as it is sometimes rather paradoxically called, the variation of constants. In more sophisticated courses the idea of the Green's function is introduced. This has many advantages, not the least of which is that the method for finding particular integrals based on such a function is much better motivated. Moreover, it is possible to fit in given boundary conditions, homogeneous or otherwise, as part of the process and this avoids artificially breaking the problem up into two parts, in the second of which a plethora of arbitrary constants is related to given boundary or initial conditions. A disadvantage of Green's function methods is that, if they are to be properly exploited, it is necessary to introduce