

# On the grasshopper problem with signed jumps

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## Abstract

The 6th problem of the 50th International Mathematical Olympiad (IMO), held in Germany, 2009, was the following. *Let  $a_1, a_2, \dots, a_n$  be distinct positive integers and let  $\mathcal{M}$  be a set of  $n - 1$  positive integers not containing  $s = a_1 + a_2 + \dots + a_n$ . A grasshopper is to jump along the real axis, starting at the point 0 and making  $n$  jumps to the right with lengths  $a_1, a_2, \dots, a_n$  in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in  $\mathcal{M}$ .* In this paper we consider a variant of the IMO problem when the numbers  $a_1, a_2, \dots, a_n$  can be negative as well. We find the sharp minimum of the cardinality of the set  $\mathcal{M}$  which blocks the grasshopper, in terms of  $n$ . In contrast with the Olympiad problem where the known solutions are purely combinatorial, for the solution of the modified problem we use the polynomial method.

## 1 Introduction

### 1.1 The original Olympiad problem

The 6th problem of the 50th International Mathematical Olympiad (IMO), held in Germany, 2009, was the following.

Let  $a_1, a_2, \dots, a_n$  be distinct positive integers and let  $\mathcal{M}$  be a set of  $n - 1$  positive integers not containing  $s = a_1 + a_2 + \dots + a_n$ . A grasshopper is to jump along the real axis, starting at the point 0 and making  $n$  jumps to the right with lengths  $a_1, a_2, \dots, a_n$  in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in  $\mathcal{M}$ .

For  $n \geq 2$  the statement of the problem is sharp. For arbitrary positive numbers  $a_1, \dots, a_n$  it is easy to find a “mine field”  $\mathcal{M}$  of  $n$  mines which makes the grasshopper’s job impossible. Such sets are, for example,  $\mathcal{M}_1 = \{a_1, a_2, \dots, a_n\}$  and  $\mathcal{M}_2 = \{s - a_1, \dots, s - a_n\}$ . In special cases more examples can be found; for example if  $(a_1, a_2, \dots, a_n) = (1, 2, \dots, n)$  then any  $n$  consecutive integers between 0 and  $s$  block the grasshopper.

The problem has been discussed in many on-line forums, as much by communities of students as by senior mathematicians; see, for example, the Art of Problem Solving / Mathlinks forum [5] or Terence Tao’s Mini-polymath project [6]. Up to now, all known solutions to the Olympiad problem are elementary and inductive.

## 1.2 Attempts to use the polynomial method

Some students at the test, who were familiar with the polynomial method, tried to apply Noga Alon's combinatorial Nullstellensatz (see Lemmas 1.1 and 1.2 in [2]). In this technique the problem is encoded via a polynomial whose nonzeros are solutions in some given domain. A powerful tool to prove the existence of a point where the polynomial does not vanish is the so-called *combinatorial Nullstellensatz*.

**Lemma 1** (Nonvanishing criterion of the combinatorial Nullstellensatz). *Let  $\mathcal{S}_1, \dots, \mathcal{S}_n$  be nonempty subsets of a field  $F$ , and let  $t_1, \dots, t_n$  be nonnegative integers such that  $t_i < |\mathcal{S}_i|$  for  $i = 1, 2, \dots, n$ . Let  $P(x_1, \dots, x_n)$  be a polynomial over  $F$  with total degree  $t_1 + \dots + t_n$ , and suppose that the coefficient of  $x_1^{t_1} x_2^{t_2} \dots x_n^{t_n}$  in  $P(x_1, \dots, x_n)$  is nonzero. Then there exist elements  $s_1 \in \mathcal{S}_1, \dots, s_n \in \mathcal{S}_n$  for which  $P(s_1, \dots, s_n) \neq 0$ .*

If we want to use the combinatorial Nullstellensatz to solve the grasshopper problem, it seems promising to choose  $\mathcal{S}_1 = \dots = \mathcal{S}_n = \{a_1, \dots, a_n\}$  and the polynomial

$$P(x_1, \dots, x_n) = V(x_1, \dots, x_n) \cdot \prod_{\ell=1}^{n-1} \prod_{m \in \mathcal{M}} ((x_1 + \dots + x_\ell) - m) \quad (1)$$

where

$$V(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i) = \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) \cdot x_{\pi(2)}^2 x_{\pi(3)}^2 \dots x_{\pi(n)}^{n-1}$$

is the so-called Vandermonde polynomial (see, for example, [1, pp. 346–347]), the symbol  $\text{Sym}(n)$  denotes the group of the permutations of the sequence  $(1, 2, \dots, n)$ , and  $\text{sgn}(\pi)$  denotes the sign of the permutation  $\pi$ .

If there exist some values  $s_1, \dots, s_n \in \{a_1, \dots, a_n\}$  such that  $P(s_1, \dots, s_n) \neq 0$ , then it follows that the grasshopper can choose the jumps  $s_1, \dots, s_n$ . The value of the Vandermonde polynomial is nonzero if and only if the jumps  $s_1, \dots, s_n$  are distinct, i.e.,  $(s_1, \dots, s_n)$  is a permutation of  $(a_1, \dots, a_n)$ , and a nonzero value of the double product on the right-hand side of (1) indicates that this order of jumps provides a safe route to the point  $s$ . (Similar polynomials appear in various applications of the combinatorial Nullstellensatz; see, for example, [3] or [4].)

The first difficulty we can recognize is that the degree is too high. To apply the combinatorial Nullstellensatz to the sets  $\mathcal{S}_1 = \dots = \mathcal{S}_n = \{a_1, \dots, a_n\}$ , we need  $\deg P = t_1 + \dots + t_n \leq n(n-1)$ , while the degree of our polynomial  $P(x_1, \dots, x_n)$  is  $\frac{1}{2}(3n-2)(n-1)$ . But there is another, less obtrusive deficiency: this approach does not use the condition that the numbers  $a_1, \dots, a_n$  are positive. We may ask: is this condition really important? What happens if this condition is omitted? In this paper we answer these questions.

## 1.3 Modified problem with signed jumps

We consider a modified problem in which the numbers  $a_1, \dots, a_n$  are allowed to be negative as well as positive. A positive value means that the grasshopper jumps to the right, while a negative value means a jump to the left. We determine the minimal size, in terms of  $n$ , of a set  $\mathcal{M}$  that blocks the grasshopper.

Optionally we may allow or prohibit the value 0 among the numbers  $a_1, \dots, a_n$ ; such a case means that the grasshopper “hops” to the same position. This option does not make a significant difference, but affects the maximal size of the set  $\mathcal{M}$ .

In order to find a sharp answer, let us collect a few simple cases when the grasshopper gets blocked. Table 1 shows three such examples. If  $n$  is odd, the size of the set  $\mathcal{M}$  is different if hops are allowed or prohibited.

Case	$\{a_1, a_2, \dots, a_n\}$	$\mathcal{M}$
$n = 2k$	$\{-k + 1, \dots, k + 1\} \setminus \{0\}$	$\{1, \dots, k + 1\}$
$n = 2k + 1$ ; “hops” allowed	$\{-k + 1, \dots, k + 1\}$	$\{1, \dots, k + 1\}$
$n = 2k + 1$ ; “hops” prohibited	$\{-k + 1, \dots, k + 2\} \setminus \{0\}$	$\{1, \dots, k + 2\}$

Table 1: Simple cases when the grasshopper is blocked.

In Section 2 we show that these examples are minimal: if the size of the set  $\mathcal{M}$  is smaller than in the examples above then the desired order of jumps exists, as stated in the next theorem, which is our main result in this paper.

**Theorem 1.** *Suppose that  $a_1, a_2, \dots, a_n$  are distinct integers and  $\mathcal{M}$  is a set of integers with  $|\mathcal{M}| \leq \left\lfloor \frac{n}{2} \right\rfloor$ . Then there exists a permutation  $(b_1, \dots, b_n)$  of the sequence  $(a_1, \dots, a_n)$  such that none of the sums  $b_1, b_1 + b_2, \dots, b_1 + b_2 + \dots + b_{n-1}$  is an element of  $\mathcal{M}$ .*

*If the numbers  $a_1, a_2, \dots, a_n$  are all nonzero, then the same holds for  $|\mathcal{M}| \leq \left\lfloor \frac{n+1}{2} \right\rfloor$  as well.*

Notice that in the modified problem the set  $\mathcal{M}$  contains fewer mines than in the statement in the Olympiad problem. This resolves the difficulties about the degree of the polynomial, allowing us to use the polynomial method.

## 2 Solution to the modified problem

In Sections 2.1 and 2.2 we prove Theorem 1 in the particular case when  $n$  is even. For the proof we use the polynomial method. The approach we follow does not apply the combinatorial Nullstellensatz directly, but there is a close connection to it; the connection between the two methods is discussed in Section 2.3. Finally, Theorem 1 is proved for odd values of  $n$  in Section 2.4.

### 2.1 Setting up the polynomials when $n$ is even

Throughout Sections 2.1–2.3 we assume that  $n = 2k$ . Since we can add arbitrary extra elements to the set  $\mathcal{M}$ , we can also assume  $|\mathcal{M}| = k$  without loss of generality.

Define the polynomial

$$Q(x_1, \dots, x_{2k}) = \sum_{\pi \in \text{Sym}(2k)} \text{sgn}(\pi) \prod_{\ell=1}^{2k-1} \prod_{m \in \mathcal{M}} ((x_{\pi(1)} + x_{\pi(2)} + \dots + x_{\pi(\ell)}) - m) \quad (2)$$

and consider the number  $Q(a_1, \dots, a_{2k})$ . If  $Q(a_1, \dots, a_{2k}) \neq 0$  then there is a permutation  $\pi \in \text{Sym}(2k)$  for which

$$\prod_{\ell=1}^{2k-1} \prod_{m \in \mathcal{M}} ((a_{\pi(1)} + a_{\pi(2)} + \dots + a_{\pi(\ell)}) - m) \neq 0.$$

This relation holds if and only if the numbers

$$a_{\pi(1)}, \quad a_{\pi(1)} + a_{\pi(2)}, \quad \dots, \quad a_{\pi(1)} + a_{\pi(2)} + \dots + a_{\pi(2k-1)}$$

all differ from the elements of  $\mathcal{M}$ . Hence, it is sufficient to prove  $Q(a_1, \dots, a_{2k}) \neq 0$ .

Since  $Q$  is an alternating polynomial, it is a multiple of the Vandermonde polynomial  $V(x_1, \dots, x_{2k})$  (we refer again to [1, pp. 346–347]). The degree of  $Q$  is at most  $k(2k-1)$ , which matches the degree of  $V$ , so

$$Q(x_1, x_2, \dots, x_{2k}) = (-1)^k c_k \cdot V(x_1, x_2, \dots, x_{2k}) \quad (3)$$

for some constant  $c_k$ . (The sign  $(-1)^k$  on the right-hand side is inserted for convenience, to make the coefficient of  $x_1^{2k-1} x_2^{2k-2} \dots x_{2k-1}$  positive.) Substituting  $a_1, \dots, a_{2k}$ , we obtain

$$Q(a_1, a_2, \dots, a_{2k}) = (-1)^k c_k \cdot V(a_1, a_2, \dots, a_{2k}).$$

Since  $a_1, \dots, a_{2k}$  are distinct,  $V(a_1, a_2, \dots, a_{2k}) \neq 0$ . So it is left to prove  $c_k \neq 0$ .

The equation (3) shows that the polynomial  $Q$  and the constant  $c_k$  do not depend on the set  $\mathcal{M}$ ; the mines are canceled out. Therefore we get the same polynomial if all mines are replaced by 0:

$$Q(x_1, \dots, x_{2k}) = \sum_{\pi \in \text{Sym}(2k)} \text{sgn}(\pi) \prod_{\ell=1}^{2k-1} (x_{\pi(1)} + \dots + x_{\pi(\ell)})^k = (-1)^k c_k \cdot V(x_1, \dots, x_{2k}).$$

Hence, the constant  $c_k$  depends only on the value of  $k$ .

Due to the complexity of the polynomial  $Q$ , it is not straightforward to access the constant  $c_k$ . The first values are listed in Table 2.

$c_1 = 1$	$c_6 = 7\,886\,133\,184\,567\,796\,056\,800$
$c_2 = 2$	$c_7 \approx 8.587 \cdot 10^{34}$
$c_3 = 90$	$c_8 \approx 4.594 \cdot 10^{51}$
$c_4 = 586\,656$	$c_9 \approx 2.060 \cdot 10^{72}$
$c_5 = 1\,915\,103\,977\,500$	$c_{10} \approx 1.237 \cdot 10^{97}$

Table 2: Values of  $c_k$  for small  $k$ .

## 2.2 Proof of $c_k \neq 0$

We prove a more general statement which contains  $c_k > 0$  as a special case.

**Definition 1.** For every  $n \geq 1$  and pair  $u, v \geq 0$  of integers, define the polynomial

$$Q^{(n,u,v)}(x_1, \dots, x_n) = \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) \left( \prod_{\ell=1}^{n-1} (x_{\pi(1)} + \dots + x_{\pi(\ell)})^u \right) (x_1 + \dots + x_n)^v. \quad (4)$$

In the case  $n = 1$ , when the product  $\prod_{\ell=1}^{n-1} (\dots)^u$  is empty, let  $Q^{(1,u,v)}(x_1) = x_1^v$ .

For every sequence  $d_1 \geq \dots \geq d_n \geq 0$  of integers, denote by  $\alpha_{d_1, \dots, d_n}^{(n,u,v)}$  the coefficient of the monomial  $x_1^{d_1} \dots x_n^{d_n}$  in  $Q^{(n,u,v)}$ . For convenience, define  $\alpha_{d_1, \dots, d_n}^{(n,u,v)} = 0$  for  $d_n < 0$  as well.

Since the polynomial  $Q^{(n,u,v)}$  is alternating, we have  $\alpha_{d_1, \dots, d_n}^{(n,u,v)} = 0$  whenever  $d_i = d_{i+1}$  for some index  $1 \leq i < n$ , and we can write

$$Q^{(n,u,v)}(x_1, \dots, x_n) = \sum_{d_1 > \dots > d_n \geq 0} \alpha_{d_1, \dots, d_n}^{(n,u,v)} \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) x_{\pi(1)}^{d_1} x_{\pi(2)}^{d_2} \cdots x_{\pi(n)}^{d_n}. \quad (5)$$

**Lemma 2.** *If  $n \geq 2$  and  $d_1 > \dots > d_n \geq 0$  then*

$$\alpha_{d_1, \dots, d_n}^{(n,u,0)} = \begin{cases} \alpha_{d_1, \dots, d_{n-1}}^{(n-1,u,u)} & \text{if } d_n = 0; \\ 0 & \text{if } d_n > 0. \end{cases}$$

*Proof.* Consider the polynomial

$$Q^{(n,u,0)}(x_1, \dots, x_n) = \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) \prod_{\ell=1}^{n-1} (x_{\pi(1)} + \dots + x_{\pi(\ell)})^u.$$

The product  $\prod_{\ell=1}^{n-1} (x_{\pi(1)} + \dots + x_{\pi(\ell)})^u$  does not contain the variable  $x_{\pi(n)}$ . Therefore the term  $x_1^{d_1} \cdots x_{n-1}^{d_{n-1}} x_n^{d_n}$  can occur only for  $d_n = 0$  and  $\pi(n) = n$ . Summing over such permutations,

$$\sum_{\pi \in \text{Sym}(n), \pi(n)=n} \text{sgn}(\pi) \prod_{\ell=1}^{n-1} (x_{\pi(1)} + \dots + x_{\pi(\ell)})^u = Q^{(n-1,u,u)}(x_1, \dots, x_{n-1}).$$

Hence, the coefficients of  $x_1^{d_1} \cdots x_{n-1}^{d_{n-1}}$  in the polynomials  $Q^{(n,u,0)}$  and  $Q^{(n-1,u,u)}$  are the same.  $\square$

**Lemma 3.** *If  $v \geq 1$  and  $d_1 > \dots > d_n \geq 0$  then*

$$\alpha_{d_1, \dots, d_n}^{(n,u,v)} = \sum_{i=1}^n \alpha_{d_1, \dots, d_{i-1}, d_i-1, d_{i+1}, \dots, d_n}^{(n,u,v-1)}.$$

*Proof.* By the definition of the polynomial  $Q^{(n,u,v)}$ , we have

$$\begin{aligned} Q^{(n,u,v)}(x_1, \dots, x_n) &= (x_1 + \dots + x_n) \cdot Q^{(n,u,v-1)}(x_1, \dots, x_n) \\ &= \sum_{i=1}^n x_i \cdot Q^{(n,u,v-1)}(x_1, \dots, x_n). \end{aligned}$$

On the left-hand side, the coefficient of  $x_1^{d_1} \cdots x_n^{d_n}$  is  $\alpha_{d_1, \dots, d_n}^{(n,u,v)}$ . On the right-hand side, the coefficient of  $x_1^{d_1} \cdots x_n^{d_n}$  in  $x_i \cdot Q^{(n,u,v-1)}$  is  $\alpha_{d_1, \dots, d_{i-1}, d_i-1, d_{i+1}, \dots, d_n}^{(n,u,v-1)}$ .  $\square$

**Lemma 4.** *If  $n \geq 1$ ,  $u, v \geq 0$ , and  $d_1 > \dots > d_n \geq 0$  then*

$$\alpha_{d_1, \dots, d_n}^{(n,u,v)} \geq 0.$$

*Proof.* Apply induction on the value of  $n(u+1) + v$ .

If  $n = 1$  then we have  $Q^{(1,u,v)}(x_1) = x_1^v$ , and the only nonzero coefficient is positive.

Now suppose that  $n \geq 2$ , and the induction hypothesis holds for all cases when  $n(u+1) + v$  is smaller. Then the induction step is provided by Lemma 2 for  $v = 0$ , and by Lemma 3 for  $v \geq 1$ .  $\square$

**Lemma 5.** Suppose that  $n \geq 1$ ,  $u \geq n/2$ ,  $v \geq 0$ , and  $d_1 > \dots > d_n \geq 0$  are integers such that  $d_1 + \dots + d_n = (n-1)u + v$  and  $d_n \leq v$ . Then

$$\alpha_{d_1, \dots, d_n}^{(n, u, v)} > 0.$$

*Proof.* Apply induction on the value of  $n(u+1) + v$ .

In case  $n = 1$  the condition  $d_1 + \dots + d_n = (n-1)u + v$  reduces to  $d_1 = v$ . Then we have  $Q^{(1, u, v)}(x_1) = x_1^v$  and thus  $\alpha_{d_1}^{(n, u, v)} = 1 > 0$ .

Now suppose that  $n \geq 2$  and the induction hypothesis holds for all cases when  $n(u+1) + v$  is smaller. We consider two cases, depending on the value of  $v$ .

*Case 1:  $v = 0$ .* From the condition  $d_n \leq v$  we get  $d_n = 0$ . Then, by Lemma 2,

$$\alpha_{d_1, \dots, d_{n-1}, 0}^{(n, u, 0)} = \alpha_{d_1, \dots, d_{n-1}}^{(n-1, u, u)}.$$

Apply the induction hypothesis to  $(n', u', v') = (n-1, u, u)$  and  $(d'_1, \dots, d'_{n-1}) = (d_1, \dots, d_{n-1})$ . The condition  $u' \geq n'/2$  is satisfied, because  $u' = u \geq n/2 > n'/2$ . Since  $v = d_n = 0$ , we have

$$d'_1 + d'_2 + \dots + d'_{n'} = d_1 + d_2 + \dots + d_{n-1} + d_n = (n-1)u = (n'-1)u' + v'.$$

Finally,

$$d'_{n'} = d_{n-1} \leq \frac{d_1 + d_2 + \dots + d_{n-1}}{n-1} = u = v'.$$

Hence, the numbers  $n', u', v'$ , and  $d'_1, \dots, d'_{n-1}$  satisfy the conditions of the lemma. By the induction hypothesis and Lemma 2 we can conclude

$$\alpha_{d_1, \dots, d_{n-1}, d_n}^{(n, u, v)} = \alpha_{d_1, \dots, d_{n-1}, 0}^{(n, u, 0)} = \alpha_{d_1, \dots, d_{n-1}}^{(n-1, u, u)} > 0.$$

*Case 2:  $v > 0$ .* By Lemma 3, we have

$$\alpha_{d_1, \dots, d_n}^{(n, u, v)} = \sum_{i=1}^n \alpha_{d_1, \dots, d_{i-1}, d_i-1, d_{i+1}, \dots, d_n}^{(n, u, v-1)}.$$

By Lemma 4, all terms are nonnegative on the right-hand side. We show that there is at least one positive term among them.

Since

$$d_1 + \dots + d_n = (n-1)u + v \geq (n-1) \cdot \frac{n}{2} + 1 > (n-1) + (n-2) + \dots + 1 + 0,$$

there exists an index  $i$ ,  $1 \leq i \leq n$ , for which  $d_i > n-i$ . Let  $i_0$  be the largest such index. If  $i_0 < n$  then  $d_{i_0} - 1 > d_{i_0+1}$ . Otherwise, if  $i_0 = n$ , we have  $d_n - 1 \geq 0$ .

Now apply the induction hypothesis to  $(n', u', v') = (n, u, v-1)$  and  $(d'_1, \dots, d'_n) = (d_1, \dots, d_{i_0-1}, d_{i_0}-1, d_{i_0+1}, \dots, d_n)$ . By the choice of  $i_0$  we have  $d'_1 > \dots > d'_n \geq 0$ . The condition  $u' \geq n'/2$  obviously satisfied. Finally,  $d'_n \leq v$  holds, too, due to  $d'_n = \max(d_n-1, 0) \leq v-1$ .

Hence, the induction hypothesis can be applied and we conclude

$$\alpha_{d_1, \dots, d_n}^{(n, u, v)} \geq \alpha_{d_1, \dots, d_{i_0-1}, d_{i_0}-1, d_{i_0+1}, \dots, d_n}^{(n, u, v-1)} > 0.$$

□

**Corollary 1.**  $c_k > 0$  for every positive integer  $k$ .

*Proof.* Apply Lemma 5 with  $n = 2k$ ,  $u = k$ ,  $v = 0$ , and  $(d_1, \dots, d_{2k}) = (2k-1, 2k-2, \dots, 1, 0)$ . The conditions of the lemma are satisfied, so

$$c_k = \alpha_{2k-1, 2k-2, \dots, 1, 0}^{(2k, k, 0)} > 0.$$

□

Corollary 1 completes the proof of Theorem 1 when  $n$  is even.

## 2.3 An alternative approach using the combinatorial Nullstellensatz

It is natural to ask whether the combinatorial Nullstellensatz is applicable to solve the modified problem, as was mentioned in the introduction. As in Section 1.1, let  $\mathcal{S}_1 = \dots = \mathcal{S}_{2k} = \{a_1, \dots, a_{2k}\}$ ,  $t_1 = \dots = t_{2k} = 2k-1$ , and consider the polynomial

$$P(x_1, \dots, x_{2k}) = V(x_1, x_2, \dots, x_{2k}) \cdot \prod_{\ell=1}^{2k-1} \prod_{m \in \mathcal{M}} ((x_1 + x_2 + \dots + x_\ell) - m).$$

The total degree of  $P$  is  $\binom{2k}{2} + (2k-1)k = 2k(2k-1)$ . To apply the nonvanishing lemma, it is sufficient to prove that the coefficient of the monomial  $x_1^{2k-1} x_2^{2k-1} \dots x_{2k}^{2k-1}$  in  $P$  is nonzero. We show this coefficient is exactly  $c_k$ .

Compare the coefficient of  $x_1^{2k-1} x_2^{2k-1} \dots x_{2k}^{2k-1}$  in the polynomials

$$\begin{aligned} P(x_1, \dots, x_{2k}) &= \sum_{\pi \in \text{Sym}(2k)} \text{sgn}(\pi) x_{\pi(2)}^2 x_{\pi(3)}^2 \dots x_{\pi(2k)}^{2k-1} \prod_{\ell=1}^{2k-1} \prod_{m \in \mathcal{M}} ((x_1 + \dots + x_\ell) - m), \\ P_1(x_1, \dots, x_{2k}) &= \sum_{\pi \in \text{Sym}(2k)} \text{sgn}(\pi) x_{\pi(2)}^2 x_{\pi(3)}^2 \dots x_{\pi(2k)}^{2k-1} \prod_{\ell=1}^{2k-1} (x_1 + \dots + x_\ell)^k, \end{aligned}$$

and

$$\begin{aligned} P_2(x_1, \dots, x_{2k}) &= \sum_{\pi \in \text{Sym}(2k)} \text{sgn}(\pi) x_2 x_3^2 \dots x_{2k}^{2k-1} \prod_{\ell=1}^{2k-1} (x_{\pi^{-1}(1)} + \dots + x_{\pi^{-1}(\ell)})^k \\ &= x_2 x_3^2 \dots x_{2k}^{2k-1} \cdot Q(x_1, \dots, x_{2k}) \\ &= c_k \cdot x_2 x_3^2 \dots x_{2k}^{2k-1} \cdot (-1)^k V(x_1, \dots, x_{2k}) \\ &= c_k \cdot x_2 x_3^2 \dots x_{2k}^{2k-1} \cdot V(x_{2k}, \dots, x_1). \end{aligned} \tag{6}$$

The difference between  $P$  and  $P_1$  is only in the constants  $m$ ; the maximal degree terms are the same.

For a fixed  $\pi \in \text{Sym}(2k)$ , the polynomials

$$x_{\pi(2)}^2 x_{\pi(3)}^2 \dots x_{\pi(2k)}^{2k-1} \prod_{\ell=1}^{2k-1} (x_1 + \dots + x_\ell)^k$$

and

$$x_2 x_3^2 \cdots x_{2k}^{2k-1} \prod_{\ell=1}^{2k-1} (x_{\pi^{-1}(1)} + \cdots + x_{\pi^{-1}(\ell)})^k$$

differ only in the order of the variables. Hence, the coefficients of the monomial  $x_1^{2k-1} x_2^{2k-1} \cdots x_{2k}^{2k-1}$  are the same in  $P$ ,  $P_1$ , and  $P_2$ . From the last line of (6) it can be seen that this coefficient is  $c_k$ .

Hence, the method of alternating sums (Section 2.1) and applying the combinatorial Nullstellensatz as above are closely related and lead to the same difficulty, i.e., to proving  $c_k \neq 0$ .

## 2.4 The case of odd $n$

Now we finish the proof of Theorem 1. Let  $n = 2k + 1$  be an odd number. The case  $n = 1$  is trivial, so assume  $k \geq 1$ .

The proof works by inserting the value 0 into, or removing it from, the list  $a_1, \dots, a_n$  of jumps and then applying Theorem 1 in the even case which is already proved.

*Case 1: The value 0 appears among  $a_1, \dots, a_{2k+1}$ .* Without loss of generality we can assume  $a_{2k+1} = 0$ . Suppose  $|\mathcal{M}| \leq \lfloor n/2 \rfloor = k$ , and apply Theorem 1 to the numbers  $a_1, \dots, a_{2k}$  and the set  $\mathcal{M}$ . The theorem provides a permutation  $(b_1, \dots, b_{2k})$  of  $(a_1, \dots, a_{2k})$  such that  $b_1, b_1 + b_2, \dots, b_1 + \cdots + b_{2k-1}$  are not elements of  $\mathcal{M}$ . Insert the value 0 somewhere in the middle, say between the first and second positions. Then  $(b_1, 0, b_2, \dots, b_{2k})$  is a permutation of  $(a_1, \dots, a_{2k}, 0)$ , with all the required properties.

*Case 2: The numbers  $a_1, \dots, a_{2k+1}$  are all nonzero.* Suppose  $|\mathcal{M}| \leq \lfloor (n+1)/2 \rfloor = k+1$ , and apply Theorem 1 to the numbers  $a_1, \dots, a_{2k+1}, a_{2k+2} = 0$  and the set  $\mathcal{M}$ . By the theorem, there is a permutation  $(b_1, \dots, b_{2k+2})$  of  $(a_1, \dots, a_{2k+1}, 0)$  such that none of  $b_1, \dots, b_1 + \cdots + b_{2k+1}$  is an element of  $\mathcal{M}$ . Deleting the value 0 from the sequence  $(b_1, \dots, b_{2k+2})$ , we obtain the permutation we want.  $\square$

## 3 Closing remarks

The high degree of  $P$  prevented the application of Alon's combinatorial Nullstellensatz from solving the original Olympiad problem. We have demonstrated that if the constraint that the jumps are positive is removed, it allows us to use the polynomial method. This also shows that the sign condition was the real reason behind the degree being too high.

### 3.1 Extension to finite fields

Finite fields and commutative groups need further consideration. The examples in Table 1 are also valid in cyclic (additive) groups, in particular in prime fields. However, our proof does not work directly for prime fields, because the constants  $c_k$  have huge prime divisors, as demonstrated below.

$$\begin{aligned} c_3 &= 2 \cdot 3^2 \cdot 5 \\ c_4 &= 2^5 \cdot 3^3 \cdot 7 \cdot 97 \\ c_5 &= 2^2 \cdot 3 \cdot 5^4 \cdot 7 \cdot 79 \cdot 103 \cdot 4483 \\ c_6 &= 2^5 \cdot 3^6 \cdot 5^2 \cdot 11 \cdot 23 \cdot 223 \cdot 239 \cdot 1\,002\,820\,739. \end{aligned}$$



So, for finite fields and groups, the problem is not yet closed.

### 3.2 Alternating sums vs. combinatorial Nullstellensatz

When we need a permutation of a finite sequence of numbers with some required property, the method applied in Section 2.1 can be a replacement for the combinatorial Nullstellensatz.

Let  $a_1, \dots, a_n$  be distinct elements of a field  $F$ , and let  $R \in F[x_1, \dots, x_n]$  be a polynomial with degree  $\binom{n}{2}$ . Suppose that we need a permutation  $(b_1, \dots, b_n)$  of  $(a_1, \dots, a_n)$  such that  $R(b_1, \dots, b_n) \neq 0$ . For such problems, it has become standard to apply the combinatorial Nullstellensatz to the polynomial  $V \cdot R$ . The method we used is different: we considered the value

$$\sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) R(a_{\pi(1)}, \dots, a_{\pi(n)}) = c \cdot V(a_1, a_2, \dots, a_n) \quad (7)$$

where the constant  $c$  is (up to the sign) the same as the coefficient of  $x_1^{n-1} \dots x_n^{n-1}$  in the polynomial  $V(x_1, \dots, x_n) \cdot R(x_1, \dots, x_n)$ , as was emphasized in Section 2.3.

Note, however, that it is easy to extend the alternating sum approach of (7) to a more general setting. By inserting an arbitrary auxiliary polynomial  $A \in F[x_1, \dots, x_n]$ , we may consider the number

$$\sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) \cdot R(a_{\pi(1)}, \dots, a_{\pi(n)}) \cdot A(a_{\pi(1)}, \dots, a_{\pi(n)}) \quad (8)$$

as well. It may happen that the sum in (8) is not 0 even when the sum of (7) vanishes.

### Appendix: A solution for the Olympiad problem

For the sake of completeness, we outline a solution for the original Olympiad problem in which all jumps were positive. We prove the following statement.

**Theorem.** *Suppose that  $a_1, a_2, \dots, a_n$  are distinct positive real numbers and let  $\mathcal{M}$  be a set of at most  $n - 1$  real numbers. Then there exists a permutation  $(i_1, \dots, i_n)$  of  $(1, 2, \dots, n)$  such that none of the numbers  $a_{i_1} + a_{i_2} + \dots + a_{i_k}$  ( $1 \leq k \leq n - 1$ ) is an element of  $\mathcal{M}$ .*

*Proof.* We employ induction on  $n$ . In the case of  $n = 1$  the statement is trivial since the set  $\mathcal{M}$  is empty; moreover there is no  $k$  with  $1 \leq k \leq n - 1$ .

Suppose that  $n \geq 2$  and the statement is true for all smaller values. Without loss of generality, we can assume that  $a_n$  is the greatest among  $a_1, a_2, \dots, a_n$ . Since we can insert extra elements into  $\mathcal{M}$ , we can also assume that  $\mathcal{M}$  has  $n - 1$  elements which are  $m_1 < \dots < m_{n-1}$ .

If  $a_1 + \dots + a_{n-1} < m_1$  then we can choose  $i_n = n$  and any order of the indices  $1, 2, \dots, n - 1$ . So we can assume  $m_1 \leq a_1 + \dots + a_{n-1}$  as well.

We consider two cases.

*Case 1:*  $a_n \in \mathcal{M}$ . Let  $\ell$  be the index for which  $a_n = m_\ell$ , and let  $\mathcal{M}' = \{m_1, \dots, m_{\ell-1}\} \cup \{m_{\ell+1} - a_n, \dots, m_{n-1} - a_n\}$ . By applying the induction hypothesis to the jumps  $a_1, \dots, a_{n-1}$  and the set  $\mathcal{M}'$ , we get a permutation  $(j_1, \dots, j_{n-1})$  of the indices  $(1, 2, \dots, n - 1)$  such that  $a_{j_1} + a_{j_2} + \dots + a_{j_k} \notin \mathcal{M}'$  for any  $k$  such that  $1 \leq k \leq n - 2$ .

Choose  $(i_1, \dots, i_n) = (j_1, n, j_2, j_3, \dots, j_{n-1})$ . We claim that this permutation of  $(1, 2, \dots, n)$  satisfies the required property.

By the choice of  $i_1$ , we have  $a_{i_1} = a_{j_1} \notin \{m_1, \dots, m_{\ell-1}\}$ . Since  $a_{i_1} < a_n = m_\ell$ , it follows that  $a_{i_1} \notin \{m_\ell, \dots, m_{n-1}\}$  and thus  $a_{i_1} \notin \mathcal{M}$ .

For  $2 \leq k \leq n-1$  we have  $a_{i_1} + \dots + a_{i_k} \geq a_{j_1} + a_n > m_\ell$ , so  $a_{i_1} + \dots + a_{i_k} \notin \{m_1, \dots, m_\ell\}$ . Moreover, since  $a_{j_1} + \dots + a_{j_{k-1}} \notin \{m_{\ell+1} - a_n, \dots, m_{n-1} - a_n\}$ , we also have  $a_{i_1} + \dots + a_{i_k} = a_{j_1} + \dots + a_{j_{k-1}} + a_n \notin \{m_{\ell+1}, \dots, m_{n-1}\}$ . Hence,  $a_{i_1} + \dots + a_{i_k} \notin \mathcal{M}$ .

*Case 2:  $a_n \notin \mathcal{M}$ .* Let  $\mathcal{M}' = \{m_2 - a_n, \dots, m_{n-1} - a_n\}$ , and apply the induction hypothesis to the jumps  $a_1, \dots, a_{n-1}$  and the set  $\mathcal{M}'$ . We get a permutation  $(j_1, \dots, j_{n-1})$  of  $(1, 2, \dots, n-1)$  such that  $a_{j_1} + a_{j_2} + \dots + a_{j_k} \notin \mathcal{M}'$  for any  $k$  with  $1 \leq k \leq n-2$ .

Let  $\ell$ ,  $1 \leq \ell \leq n-1$ , be the first index for which  $a_{j_1} + \dots + a_{j_\ell} \geq m_1$ . (By the assumption  $m_1 \leq a_1 + \dots + a_{n-1}$ , there exists such an index.) Then choose  $(i_1, \dots, i_n) = (j_1, \dots, j_{\ell-1}, n, j_\ell, \dots, j_{n-1})$ . (If  $\ell = 1$ , choose  $(n, j_1, \dots, j_{n-1})$ .) We show that this permutation fulfills the desired property.

For  $1 \leq k \leq \ell-1$  we have  $a_{i_1} + \dots + a_{i_k} \leq a_{j_1} + \dots + a_{j_{\ell-1}} < m_1$ , so  $a_{i_1} + \dots + a_{i_k} \notin \mathcal{M}$ .

For  $\ell \leq k \leq n-1$  we have  $a_{i_1} + \dots + a_{i_k} \geq a_{j_1} + \dots + a_{j_{\ell-1}} + a_n > a_{j_1} + \dots + a_{j_{\ell-1}} + a_{j_\ell} \geq m_1$ . Since  $a_{j_1} + \dots + a_{j_{k-1}} \notin \mathcal{M}'$ , it follows that  $a_{i_1} + \dots + a_{i_k} = a_{j_1} + \dots + a_{j_{k-1}} + a_n \notin \{m_2, \dots, m_{n-1}\}$ . Hence,  $a_{i_1} + \dots + a_{i_k} \notin \mathcal{M}$ .  $\square$

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